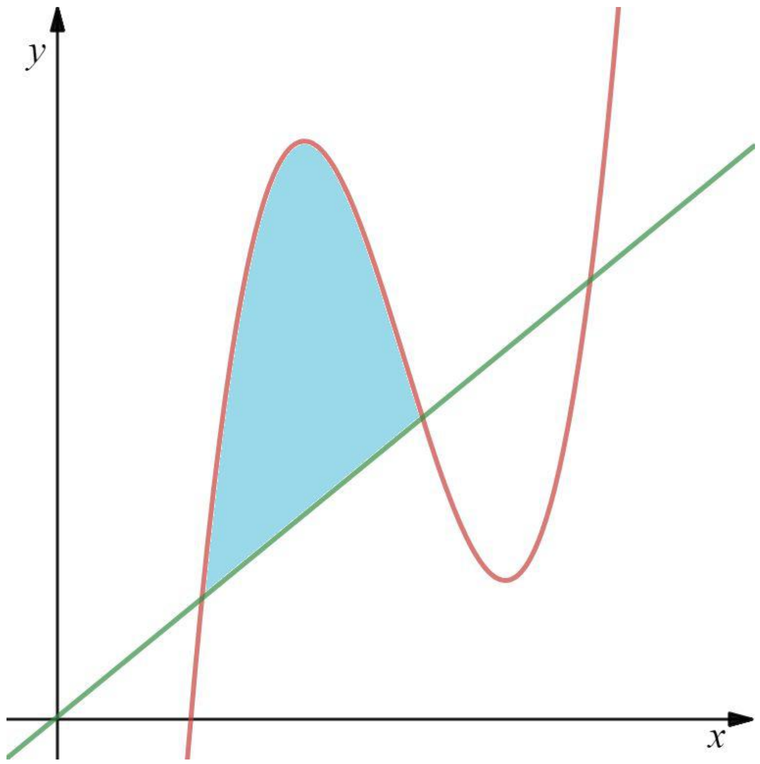
Pure Mathematics:
Algebra, Functions & Coordinate Geometry



Material Covered

Algebra

1. Algebraic Expressions & The Factor Theorem
2. Quadratic Equations
3. Inequalities

Algebraic Graphs

1. Graphs of Functions
2. Transformations

Coordinate Geometry

1. Equation of a Straight Line
2. The Coordinate Geometry of a Circle

Algebra

$$a^2 + b^2 \equiv c^2$$

Specification Points - AQA

	Content
B1	Understand and use the laws of indices for all rational exponents.

	Content
B2	Use and manipulate surds, including rationalising the denominator.

	Content
B3	Work with quadratic functions and their graphs; the discriminant of a quadratic function, including the conditions for real and repeated roots; completing the square; solution of quadratic equations including solving quadratic equations in a function of the unknown.

	Content
B4	Solve simultaneous equations in two variables by elimination and by substitution, including one linear and one quadratic equation.

	Content
B5	Solve linear and quadratic inequalities in a single variable and interpret such inequalities graphically, including inequalities with brackets and fractions. Express solutions through correct use of 'and' and 'or', or through set notation. Represent linear and quadratic inequalities such as $y > x + 1$ and $y > ax^2 + bx + c$ graphically.

	Content
B6	Manipulate polynomials algebraically, including expanding brackets and collecting like terms, factorisation and simple algebraic division; use of the factor theorem. Simplify rational expressions including by factorising and cancelling, and algebraic division (by linear expressions only).

Specification Points – OCR A

OCR Ref.	Subject Content	Stage 1 learners should ...
1.02d	Quadratic functions	d) Be able to work with quadratic functions and their graphs, and the discriminant (D or Δ) of a quadratic function, including the conditions for real and repeated roots. *i.e. Use the conditions:* *1.* $b^2 - 4ac > 0 \Rightarrow$ *real distinct roots* *2.* $b^2 - 4ac = 0 \Rightarrow$ *repeated roots* *3.* $b^2 - 4ac < 0 \Rightarrow$ *roots are not real* *to determine the number and nature of the roots of a quadratic equation and relate the results to a graph of the quadratic function.*
1.02e		e) Be able to complete the square of the quadratic polynomial $ax^2 + bx + c$. *e.g. Writing* $y = ax^2 + bx + c$ *in the form* $y = a(x+p)^2 + q$ *in order to find the line of symmetry* $x = -p$*, the turning point* $(-p, q)$ *and to determine the nature of the roots of the equation* $ax^2 + bx + c = 0$ *for example* $2(x+3)^2 + 4 = 0$ *has no real roots because* $4 > 0$.
1.02f		f) Be able to solve quadratic equations including quadratic equations in a function of the unknown. *e.g.* $x^4 - 5x^2 + 6 = 0$, $x^{\frac{2}{3}} - 5x^{\frac{1}{3}} + 4 = 0$ *or* $\frac{5}{(2x-1)^2} - \frac{10}{2x-1} = 1$.

OCR Ref.	Subject Content	Stage 1 learners should ...
1.02g	Inequalities	g) Be able to solve linear and quadratic inequalities in a single variable and interpret such inequalities graphically, including inequalities with brackets and fractions. *e.g.* $10 < 3x + 1 < 16$, $(2x+5)(x+3) > 0$. *[Quadratic equations with complex roots are excluded.]*
1.02h		h) Be able to express solutions through correct use of 'and' and 'or', or through set notation. *Familiarity is expected with the correct use of set notation for intervals, e.g.* $\{x : x > 3\}$, $\{x : -2 \le x \le 4\}$, $\{x : x > 3\} \cup \{x : -2 \le x \le 4\}$, $\{x : x > 3\} \cap \{x : -2 \le x \le 4\}$, $\varnothing$. *Familiarity is expected with interval notation, e.g.* $(2, 3)$, $[2, 3)$ *and* $[2, \infty)$.
1.02i		i) Be able to represent linear and quadratic inequalities such as $y > x + 1$ and $y > ax^2 + bx + c$ graphically.

OCR Ref.	Subject Content	Stage 1 learners should ...
1.02j 1.02k	Polynomials	j) Be able to manipulate polynomials algebraically. *Includes expanding brackets, collecting like terms, factorising, simple algebraic division and use of the factor theorem.* *Learners should be familiar with the terms "quadratic", "cubic" and "parabola".* *Learners should be familiar with the factor theorem as:* *1.* $f(a) = 0 \Leftrightarrow (x - a)$ *is a factor of* $f(x)$; *2.* $f(\frac{b}{a}) = 0 \Leftrightarrow (ax - b)$ *is a factor of* $f(x)$. *They should be able to use the factor theorem to find a linear factor of a polynomial normally of degree* ≤ 3*. They may also be required to find factors of a polynomial, using any valid method, e.g. by inspection.*

Specification Points – OCR MEI

Ma2	Be able to solve quadratic equations.	By factorising, completing the square, using the formula and graphically. Includes quadratic equations in a function of the unknown.
a3	Be able to find the discriminant of a quadratic function and understand its significance.	The condition for distinct real roots of $ax^2 + bx + c = 0$ is: Discriminant > 0. The condition for repeated roots is: Discriminant $= 0$. The condition for no real roots is: Discriminant < 0.
a4	Be able to solve linear simultaneous equations in two unknowns.	By elimination and by substitution.
a5	Be able to solve simultaneous equations in two unknowns with one equation linear and one quadratic.	By elimination and by substitution.
a6	Know the significance of points of intersection of two graphs with relation to the solution of equations.	Including simultaneous equations.

Ma7	Be able to solve linear inequalities in one variable. Be able to represent and interpret linear inequalities graphically e.g. $y > x + 1$.	Including those containing brackets and fractions.
a8	Be able to solve quadratic inequalities in one variable. Be able to represent and interpret quadratic inequalities graphically e.g. $y > ax^2 + bx + c$.	Algebraic and graphical treatment of solution of quadratic inequalities. For regions defined by inequalities learners must state clearly which regions are included and whether the boundaries are included. No particular shading convention is expected.

Mf1	Be able to add, subtract, multiply and divide polynomials.	Expanding brackets and collecting like terms.
f2	Understand the factor theorem and be able to use it to factorise a polynomial or to determine its zeros.	$f(a) = 0 \Leftrightarrow (x - a)$ is a factor of $f(x)$. Including when solving a polynomial equation.

Specification Points - Edexcel

2.3	**Work with quadratic functions and their graphs.**	**The notation $f(x)$ may be used**
	The discriminant of a quadratic function, including the conditions for real and repeated roots.	**Need to know and to use** $b^2 - 4ac > 0$, $b^2 - 4ac = 0$ **and** $b^2 - 4ac < 0$
	Completing the square.	$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right)$
	Solution of quadratic equations	**Solution of quadratic equations by factorisation, use of the formula, use of a calculator and completing the square.**
	including solving quadratic equations in a function of the unknown.	**These functions could include powers of x, trigonometric functions of x, exponential and logarithmic functions of x.**

2 **Algebra and functions** *continued*	2.5	**Solve linear and quadratic inequalities in a single variable and interpret such inequalities graphically,**	**e.g. solving** $ax + b > cx + d$, $px^2 + qx + r \geq 0$, $px^2 + qx + r < ax + b$ **and interpreting the third inequality as the range of x for which the curve $y = px^2 + qx + r$ is below the line with equation $y = ax + b$**
		including inequalities with brackets and fractions.	**These would be reducible to linear or quadratic inequalities** **e.g.** $\frac{a}{x} < b$ **becomes** $ax < bx^2$
		Express solutions through correct use of 'and' and 'or', or through set notation.	**So, e.g.** $x < a$ **or** $x > b$ **is equivalent to** $\{x : x < a\} \cup \{x : x > b\}$ **and** $\{x : c < x\} \cap \{x : x < d\}$ **is equivalent to** $x > c$ **and** $x < d$
		Represent linear and quadratic inequalities such as $y > x + 1$ and $y > ax^2 + bx + c$ graphically.	**Shading and use of dotted and solid line convention is required.**
	2.6	**Manipulate polynomials algebraically, including expanding brackets and collecting like terms, factorisation and simple algebraic division; use of the factor theorem.**	**Only division by $(ax + b)$ or $(ax - b)$ will be required. Students should know that if $f(x) = 0$ when $x = a$, then $(x - a)$ is a factor of $f(x)$.** **Students may be required to factorise cubic expressions such as** $x^3 + 3x^2 - 4$ **and** $6x^3 + 11x^2 - x - 6$.
		Simplify rational expressions, including by factorising and cancelling, and algebraic division (by linear expressions only).	Denominators of rational expressions will be linear or quadratic, e.g. $\frac{1}{ax+b}$, $\frac{ax+b}{px^2+qx+r}$, $\frac{x^3+a^3}{x^2-a^2}$

Algebraic Expressions

We can **manipulate algebraic expressions** in a **number** of **ways**:

- **Factorising**: $x^4 - 4$
- **Expanding**: $(x^2 + 3)(x - 2)$
- **Rationalising denominator**: $\dfrac{x}{1 + \sqrt{x}}$

- Usually we want to **give** these **expressions** in their **simplest form**.
- This involves **factorising**, **rationalising the denominator** and **cancelling**.

Factor Theorem

We can **apply** the **factor theorem** to find **factors** of **complex algebraic expressions**:

For a **polynomial** $f(x)$:

- $(x-a)$ is a **factor** $\Leftrightarrow f(a) = \mathbf{0}$
- $(ax+b)$ is a **factor** $\Leftrightarrow f\left(-\frac{b}{a}\right) = \mathbf{0}$

- Once we have **confirmed** that $(x-a)$ is a **factor** we can use **polynomial long division** to **factorise**.

Exemplar Exam Question

Remember **factor theorem** and how to use it

Cubic function, needs more work to factorise

1) Given $f(x) = 6x^3 - 13x^2 + 4$, use factor theorem to show that $(2x + 1)$ is a factor of $f(x)$. Hence, factorise $f(x)$ completely.

Can use **polynomial long division** to divide $f(x)$

[5 Marks]

Time limit: about 5 minutes, around 5 key steps

Exemplar Exam Question Answer

$$f(x) = 6x^3 - 13x^2 + 4$$

Use factor theorem to show $(2x+1)$ is a factor of $f(x)$

Factor theorem states that $(2x+1)$ is factor if $f\left(-\frac{1}{2}\right) = 0$

$$f\left(-\frac{1}{2}\right) = 6\left(-\frac{1}{2}\right)^3 - 13\left(-\frac{1}{2}\right)^2 + 4$$

$$= -\frac{6}{8} - \frac{13}{4} + 4$$

$$= 0$$

[1 Mark]

Exemplar Exam Question Answer

$$f(x) = 6x^3 - 13x^2 + 4$$

Divide $f(x)$ by $(2x + 1)$

Use polynomial long division

$$(2x + 1)\overline{)6x^3 - 13x^2 + 0x + 4}$$

Exemplar Exam Question Answer

$$f(x) = 6x^3 - 13x^2 + 4$$

$$
\begin{array}{r}
3x^2 - 8x + 4 \\
(2x+1)\overline{\big)\, 6x^3 - 13x^2 + 0x + 4} \\
\underline{6x^3 + 3x^2 } \\
-16x^2 + 0x \\
\underline{-16x^2 - 8x } \\
8x + 4 \\
\underline{8x + 4} \\
0
\end{array}
$$

[2 Marks]

Exemplar Exam Question Answer

$$f(x) = (2x+1)\left(3x^2 - 8x + 4\right)$$

Factorise remaining quadratic

$$3x^2 - 8x + 4 = 3x^2 - 6x - 2x + 4$$

$$= 3x(x-2) - 2(x-2) = (3x-2)(x-2)$$ **[1 Mark]**

Write full factorised function

$$f(x) = (2x+1)(3x-2)(x-2)$$ **[1 Mark]**

Quadratics

Quadratic equations can be **solved** by **factorising**:

$$ax^2 + bx + c = 0 \longrightarrow (mx + p)(nx + q) = 0$$

… or by **completing the square**:

$$ax^2 + bx + c = 0 \longrightarrow a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right) = 0$$

Quadratics

… or by **using the quadratic formula**:

$$ax^2 + bx + c = 0 \longrightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

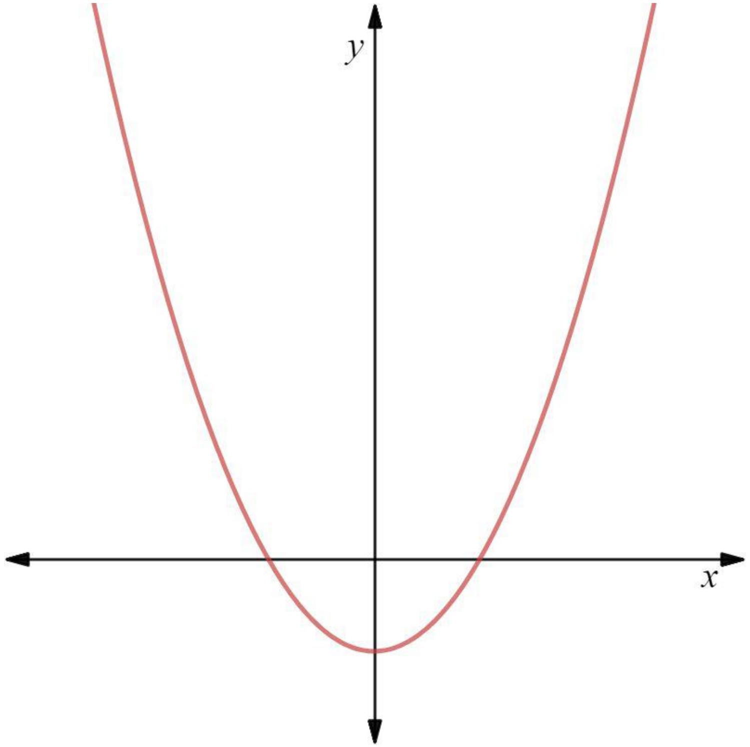
The **number of roots** for x is determined by the **discriminant** (Δ) of the **quadratic function**:

$$\Delta = b^2 - 4ac$$

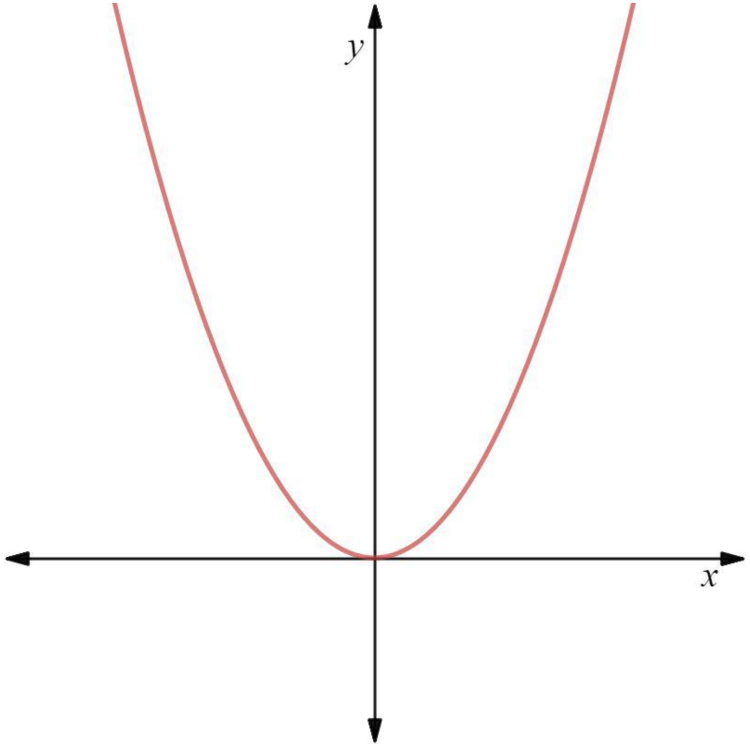
The Quadratic Discriminant

The **number** of **real**, **distinct roots** of $ax^2 + bx + c = 0$ corresponds to the **number** of **times** that the **graph** of $y = ax^2 + bx + c$ **crosses** the x-axis.

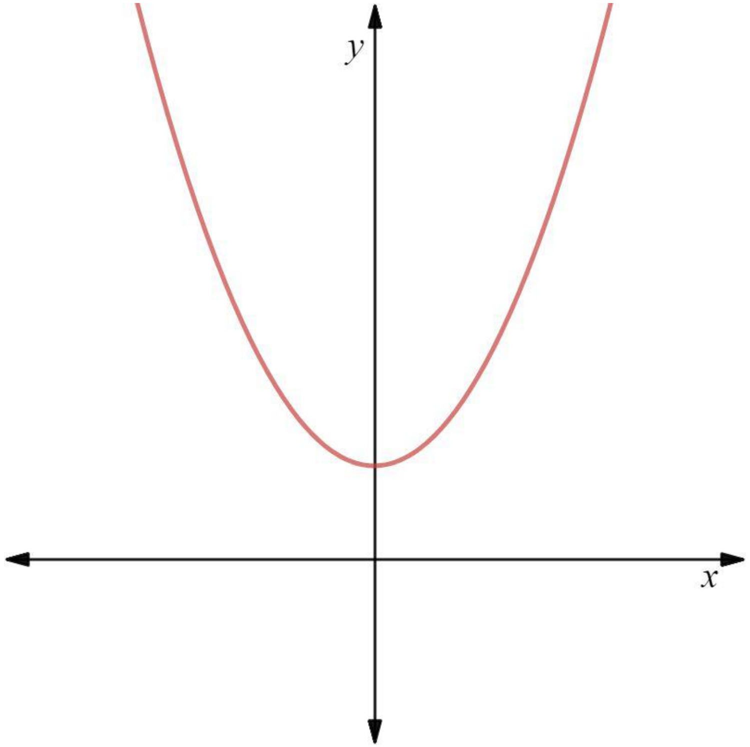
$b^2 - 4ac > 0$



$b^2 - 4ac = 0$



$b^2 - 4ac < 0$



Exemplar Exam question

- Looking for a **quadratic** and **discriminant**

1) Determine the number of roots of the following equation:

$$3x^2 + kx + \frac{1}{2} = x^2 - kx$$

Looks like a **quadratic**, but needs **rearranging.**

where $k^2 < 1$. You do not need to solve the equation.

[4 Marks]

Will **definitely come up at some point**

Equation probably isn't solvable without knowing k.

Time limit: about 4 minutes, around 4 key steps

Exemplar Exam Question Answer

Rearrange into a quadratic equation

$$3x^2 + kx + \frac{1}{2} = x^2 - kx$$

$$2x^2 + 2kx + \frac{1}{2} = 0$$

[1 Mark]

Derive discriminant

For quadratic $ax^2 + bx + c = 0$, discriminant given by $\Delta = b^2 - 4ac$

$$\Delta = (2k)^2 - 4(2)\left(\frac{1}{2}\right)$$

$$= 4k^2 - 4$$

[1 Mark]

Exemplar Exam Question Answer

Determine sign of discriminant

$$\Delta = 4(k^2 - 1)$$

Know $k^2 < 1$, so $(k^2 - 1) < 0$

Hence $\Delta < 0$

[1 Mark]

Interpret result

Δ has no real roots

[1 Mark]

Inequalities

Inequalities define a **range of values** for **variables** and **functions**.

- The **solutions** of **inequalities** will also be **inequalities**.

$$ax^2 + bx + c > 0 \longrightarrow x > p \quad \text{and} \quad x < q$$

- When **multiplying** or **dividing** both **sides** of an **inequality** by a **negative number** the **inequality sign** is **reversed**.

$$ax^2 + bx + c > 0 \longrightarrow -ax^2 - bx - c < 0$$

Inequalities

Quadratic inequalities can be **solved** in the **same way** as **quadratic equations:**

$$ax^2 + bx + c > 0$$

$$(x - p)(x - q) > 0$$

$x >$ or $< p$ and $x >$ or $< q$

Inequalities

- To **determine** which **inequality sign** is **correct** for each **root substitute** a **value** which is $>$ or $<$ p or q into $ax^2 + bx + c > 0$.
- If the **inequality** is **true** for this **value** then the selected **inequality root** is **correct**.

Exemplar Exam Question

Not solve, we'll need to **define an inequality**

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1) By plotting a quadratic function, find the set of values of x for which

$$2x^2 - 3x + 8 \leq 20 + 2x.$$

4 steps to question, between solving quadratic and interpreting inequality

Solving quadratic inequalities, need to **find roots** and **sketch graph**

[4 marks]

Exemplar Exam Question Answer

Move all terms to one side

$$2x^2 - 3x + 8 \leq 20 + 2x$$

$$2x^2 - 5x - 12 \leq 0$$

[1 Mark]

Factorise and find roots

$$(2x + 3)(x - 4) \leq 0$$

Roots are $x = -\frac{3}{2}$ or 4

[1 Mark]

Exemplar Exam Question Answer

$2x^2 - 5x - 12 \leq 0$

Sketch graph of function to determine where inequality is true

$y = 2x^2 - 5x - 12 \leq 0$

x^2 term is positive therefore graph is concave upwards (U-shaped).

$x = 0 \Rightarrow y = 2 \times 0^2 - 5 \times 0 - 12 = -12$

y-intercept at $y = -12$.

x-intercepts at $x = -\frac{3}{2}$ and 4. **[1 Mark]**

Exemplar Exam Question Answer

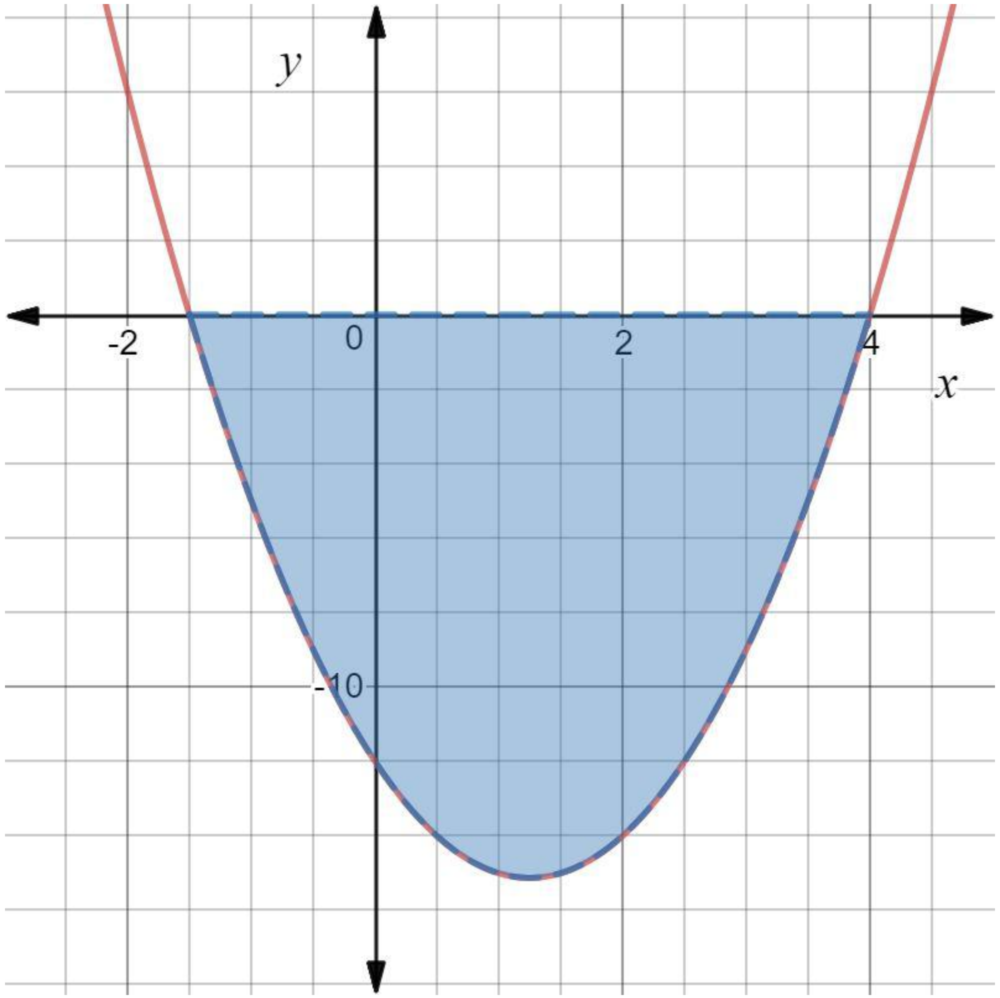
$$y = 2x^2 - 5x - 12 \leq 0$$

Determine final inequality for x

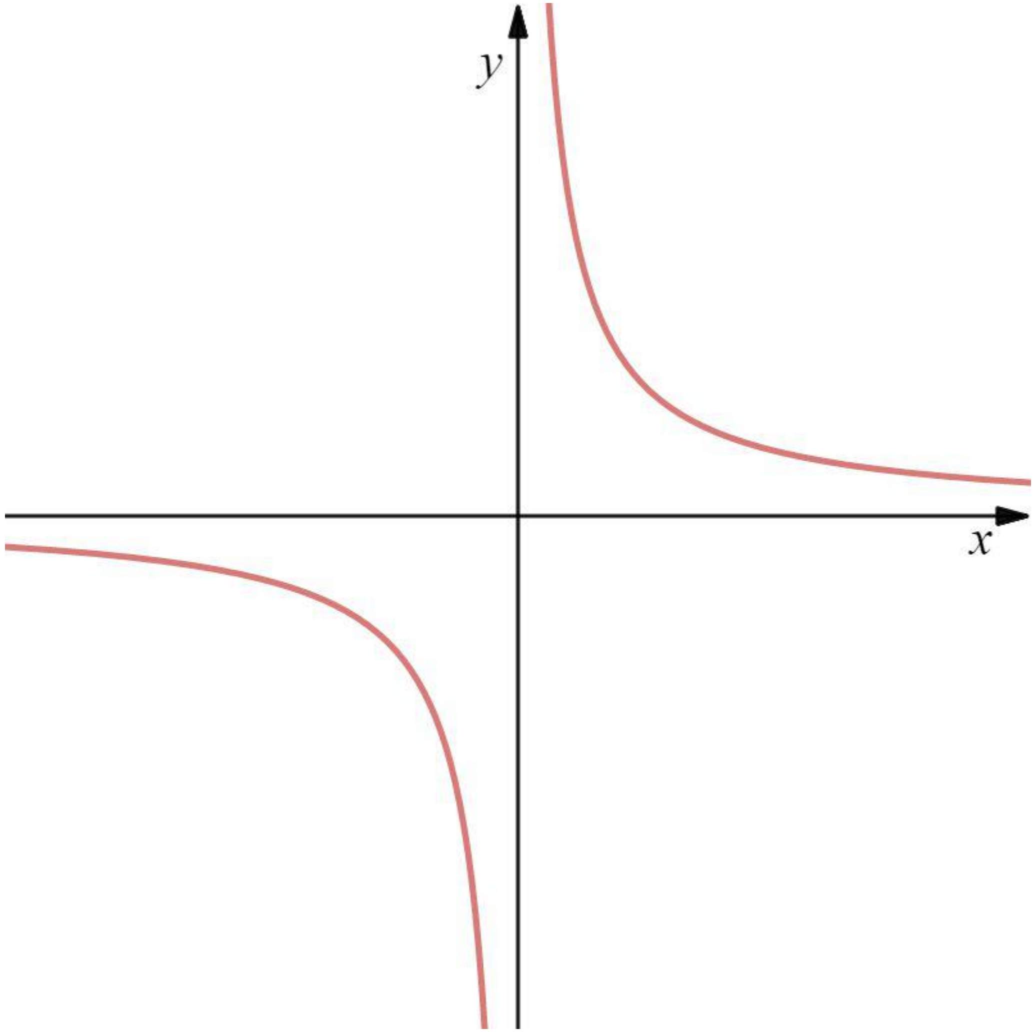
Inequality is true when $y \leq 0$ therefore when graph is below x-axis.

Inequality is true for $-\frac{3}{2} \leq x \leq 4$

[1 Mark]



Algebraic Graphs



Specification Points - AQA

	Content
B7	Understand and use graphs of functions; sketch curves defined by simple equations including polynomials, the modulus of a linear function, $y = \frac{a}{x}$ and $y = \frac{a}{x^2}$ (including their vertical and horizontal asymptotes); interpret algebraic solution of equations graphically; use intersection points of graphs to solve equations. Understand and use proportional relationships and their graphs.

	Content
B8	Understand and use composite functions; inverse functions and their graphs.

	Content
B9	Understand the effect of simple transformations on the graph of $y = f(x)$ including sketching associated graphs: $y = af(x)$, $y = f(x) + a$, $y = f(x + a)$, $y = f(ax)$, and combinations of these transformations.

Specification Points – OCR A

OCR Ref.	Subject Content	Stage 1 learners should ...
1.02m 1.02s	Curve sketching	m) Understand and be able to use graphs of functions. *The difference between plotting and sketching a curve should be known. See Section 2b.*
1.02n		n) Be able to sketch curves defined by simple equations including polynomials. *e.g. Familiarity is expected with sketching a polynomial of*
1.02t		*degree ≤ 4 in factorised form, including repeated roots.* *Sketches may require the determination of stationary points and, where applicable, distinguishing between them.*
1.02o		o) Be able to sketch curves defined by $y = \frac{a}{x}$ and $y = \frac{a}{x^2}$ (including their vertical and horizontal asymptotes).
1.02p		p) Be able to interpret the algebraic solution of equations graphically.
1.02q		q) Be able to use intersection points of graphs to solve equations. *Intersection points may be between two curves one or more of which may be a polynomial, a trigonometric, an exponential or a reciprocal graph.*

1.02w 1.02x	Graph transformations	w) Understand the effect of simple transformations on the graph of $y = f(x)$ including sketching associated graphs, describing transformations and finding relevant equations: $y = af(x)$, $y = f(x) + a$, $y = f(x + a)$ and $y = f(ax)$, for any real a.

Specification Points – OCR MEI

MC1	Understand and use graphs of functions.	
C2	Understand how to find intersection points of a curve with coordinate axes.	Including relating this to the solution of an equation.
C3	Understand and be able to use the method of completing the square to find the line of symmetry and turning point of the graph of a quadratic function and to sketch a quadratic curve (parabola).	The curve $y = a(x+p)^2 + q$ has • a minimum at $(-p,\ q)$ for $a > 0$ or a maximum at $(-p,\ q)$ for $a < 0$ • a line of symmetry $x = -p$.
C4	Be able to sketch and interpret the graphs of simple functions including polynomials.	Including cases of repeated roots for polynomials.
C5	Be able to use stationary points when curve sketching.	Including distinguishing between maximum and minimum turning points.
C6	Be able to sketch and interpret the graphs of $y = \frac{a}{x}$ and $y = \frac{a}{x^2}$.	Including their vertical and horizontal asymptotes and recognising them as graphs of proportional relationships.

MC7	Be able to sketch curves of the forms $y = af(x),\ y = f(x) + a,\ y = f(x+a)$ and $y = f(ax)$, given the curve of $y = f(x)$ and describe the associated transformations. Be able to form the equation of a graph following a single transformation.
	PURE MATH
C8	Understand the effect of combined transformations on a graph and be able to form the equation of the new graph and to sketch it. Be able to recognise the transformations that have been applied to a graph from the graph or its equation.
C9	Be able to use stationary points of inflection when curve sketching.

Mg8	Be able to find the point(s) of intersection of a line and a curve or of two curves.

Specification Points - Edexcel

Topics	What students need to learn:		
	Content		Guidance
2 **Algebra and functions** *continued*	2.7	**Understand and use graphs of functions; sketch curves defined by simple equations including polynomials** The modulus of a linear function.	**Graph to include simple cubic and quartic functions,** **e.g. sketch the graph with equation $y = x^2(2x - 1)^2$** Students should be able to sketch the graphs of $y = \lvert ax + b \rvert$ They should be able to use their graph. For example, sketch the graph with equation $y = \lvert 2x - 1 \rvert$ and use the graph to solve the equation $\lvert 2x - 1 \rvert = x$ or the inequality $\lvert 2x - 1 \rvert > x$
		$y = \frac{a}{x}$ **and** $y = \frac{a}{x^2}$ **(including their vertical and horizontal asymptotes)**	**The asymptotes will be parallel to the axes e.g. the asymptotes of the curve with equation $y = \frac{2}{x+a} + b$ are the lines with equations $y = b$ and $x = -a$**
		Interpret algebraic solution of equations graphically; use intersection points of graphs to solve equations.	

2.9	**Understand the effect of simple transformations on the graph of $y = f(x)$, including sketching associated graphs:** $y = af(x)$, $y = f(x) + a$, $y = f(x + a)$, $y = f(ax)$ and combinations of these transformations	Students should be able to find the graphs of $y = \lvert f(x) \rvert$ and $y = \lvert f(-x) \rvert$, given the graph of $y = f(x)$. Students should be able to apply a combination of these transformations to any of the functions in the A Level specification (quadratics, cubics, quartics, reciprocal, $\frac{a}{x^2}$, $\lvert x \rvert$, $\sin x$, $\cos x$, $\tan x$, e^x and a^x) and sketch the resulting graph. Given the graph of $y = f(x)$, students should be able to sketch the graph of, e.g. $y = 2f(3x)$, or $y = f(-x) + 1$, and should be able to sketch (for example) $y = 3 + \sin 2x$, $y = -\cos\left(x + \frac{\pi}{4}\right)$

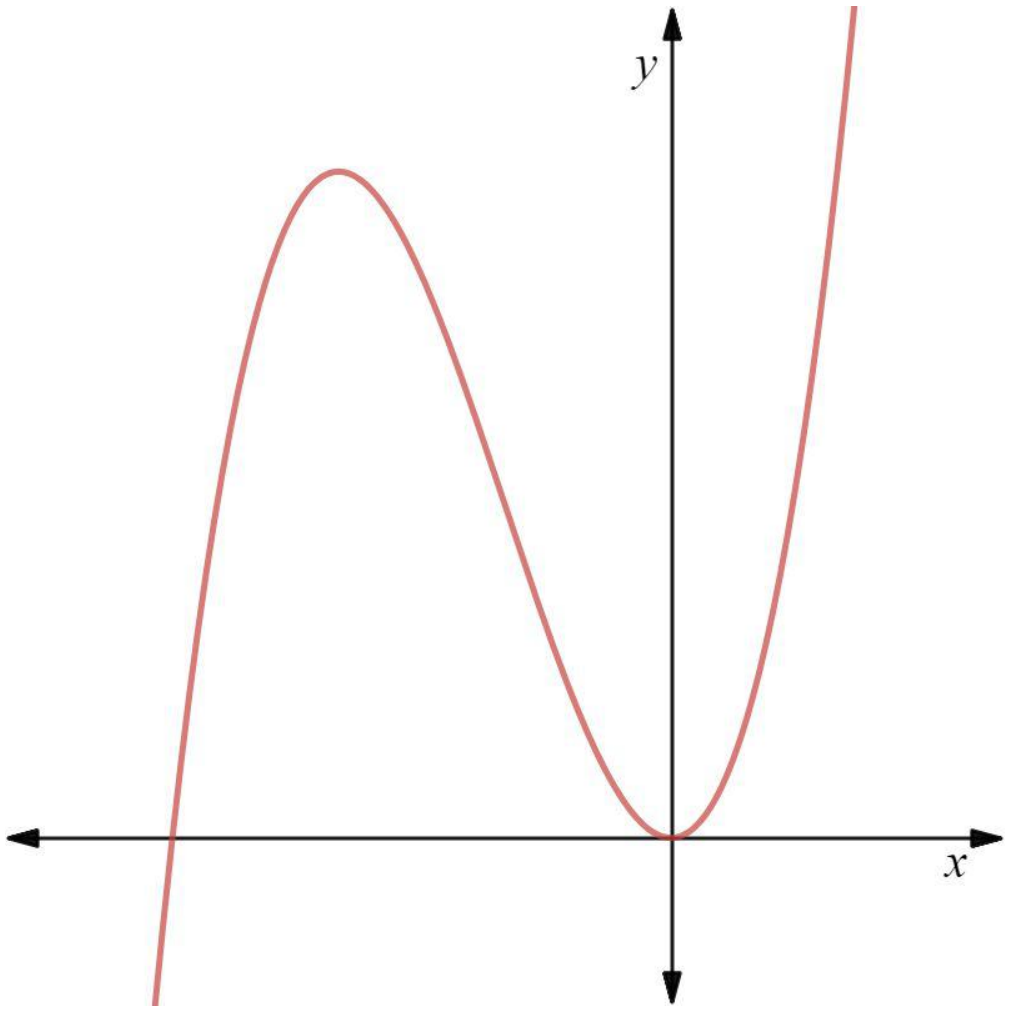
Graphs of Functions

Cubic graphs result from **plotting functions** of the **form**: $y = ax^3 + bx^2 + cx + d$

- Find y-intercept by **setting** $x = 0$.
- Find x-intercept by **setting** $y = 0$ and finding x **roots**.
- Find **turning points** by **setting** $\frac{dy}{dx} = 0$.

Graphs of Functions

$$y = x^3 + 3x^2$$

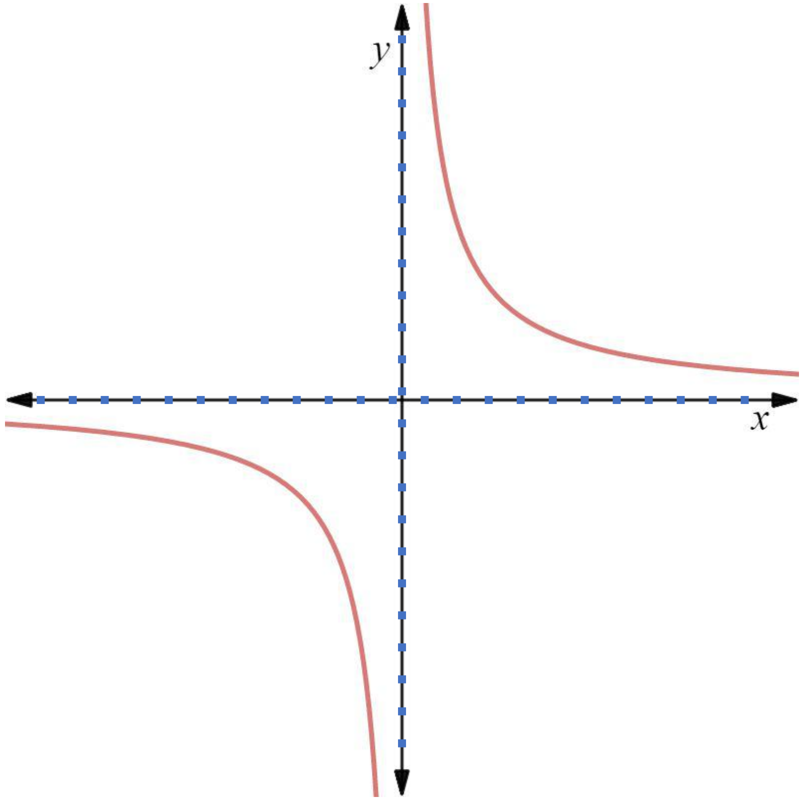


Graphs of Functions

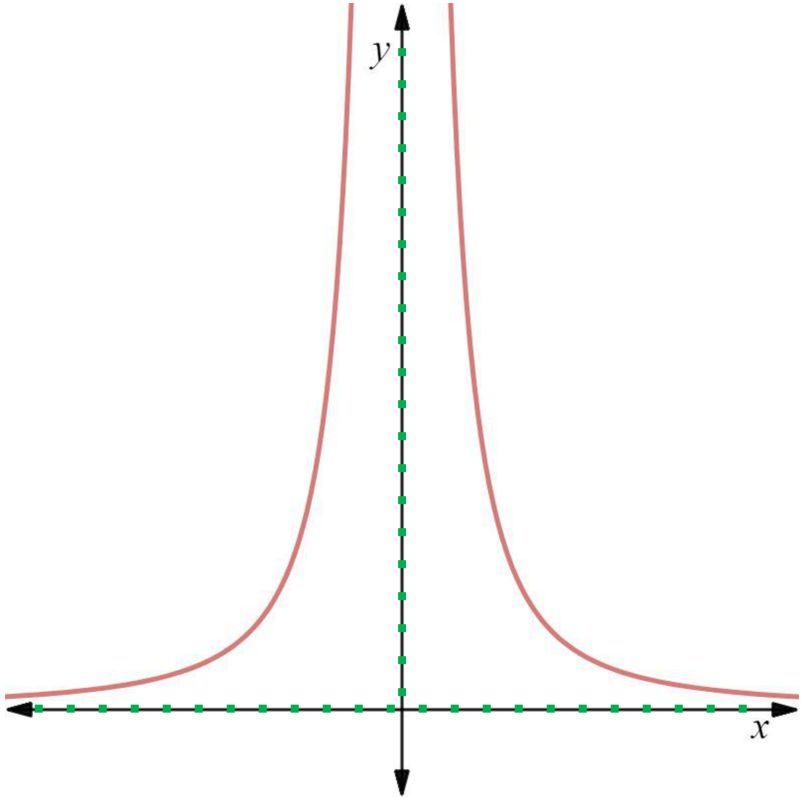
Negative index curves have **asymptotes** where the **denominator** $= \mathbf{0}$.

- **Curves tend** towards **asymptotes** but never **touch** them.

$$y = \frac{1}{x}$$



$$y = \frac{1}{x^2}$$



Exemplar Exam Question

Cubic graph, **two turning points**, **three** potential **roots**

Need to determine **shape** and **intercepts** of graph

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Must **calculate key co-ordinates**

1) Sketch the graph of $\boldsymbol{y} = (\boldsymbol{x}+\mathbf{1})(\boldsymbol{x}-\mathbf{1})(\boldsymbol{x}+\mathbf{1})$, clearly showing any intercepts with the axes, and the $\boldsymbol{x}$-co-ordinates of the stationary points.

[4 marks]

What we need to **calculate** from **function**

Exemplar Exam Question Answer

$y = (x+1)(x-1)(x+1)$

Calculate x-axis intercept

x-axis intercepts occur when $y = 0$. Need to determine roots of function.

Function already factorised so can simply read off roots

Graph has a single root at $x = 1$ and a repeated root at $x = -1$

[1 Mark]

Exemplar Exam Question Answer

$y = (x+1)(x-1)(x+1)$

Calculate y-axis intercept

Let $x = 0$

$$y = (x+1)(x-1)(x+1)$$
$$= (0+1)(0-1)(0+1)$$
$$= -1$$

y-intercept at $y = -1$

[1 Mark]

Exemplar Exam Question Answer

Calculate turning points

Expand out brackets

$$y = (x + 1)(x - 1)(x + 1)$$

$$= (x + 1)(x^2 - 1)$$

$$= x^3 + x^2 - x - 1$$

Exemplar Exam Question Answer

Differentiate expanded equation

$$y = x^3 + x^2 - x - 1$$

$$\frac{dy}{dx} = 3x^2 + 2x - 1$$

Set $\frac{dy}{dx} = 0$ and solve

$$0 = 3x^2 + 2x - 1$$

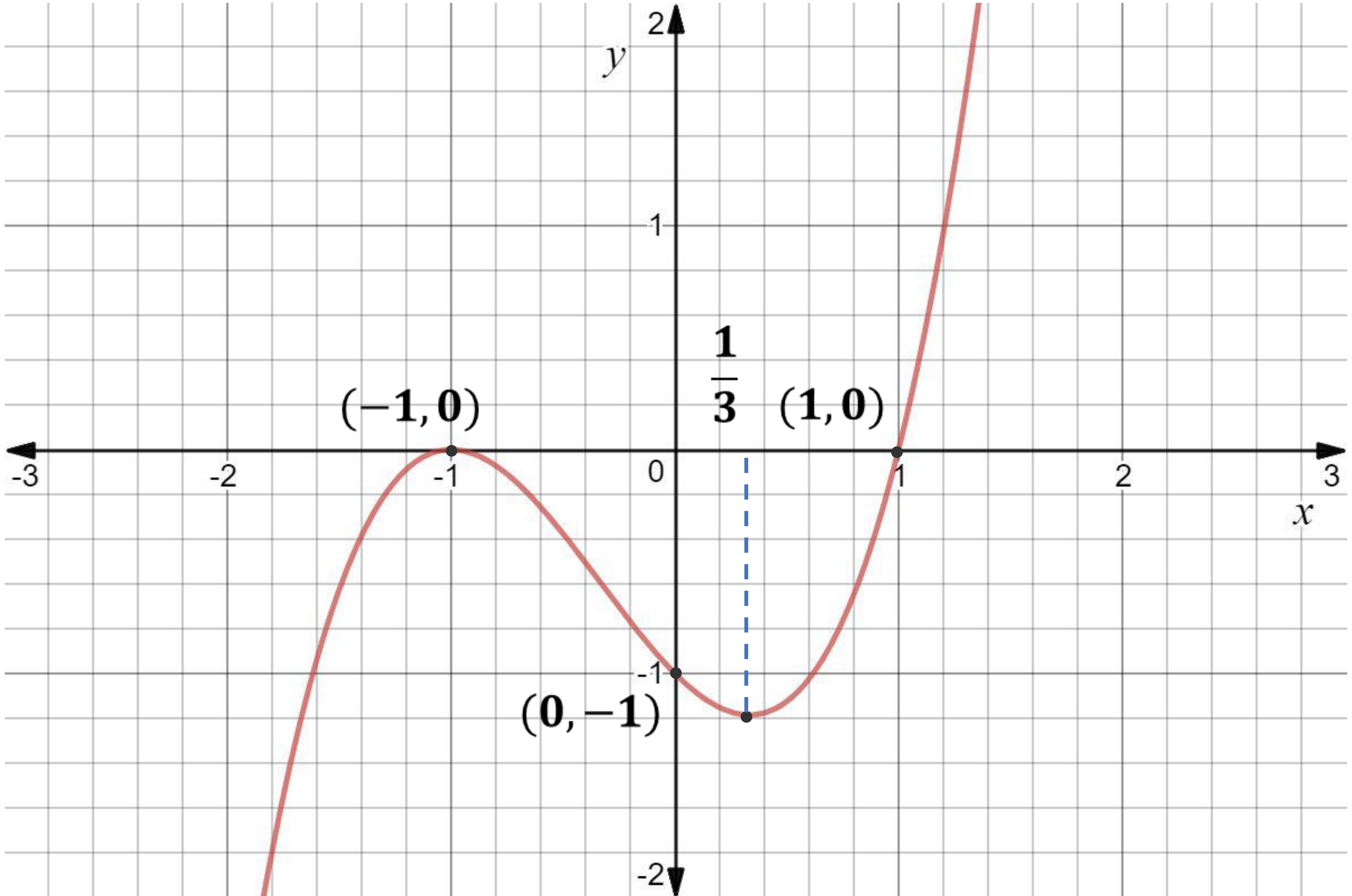
$$0 = (3x - 1)(x + 1)$$

Turning points at $x = \frac{1}{3}$ or -1

[1 Mark]

Exemplar Exam Question Answer

Mark points and draw graph



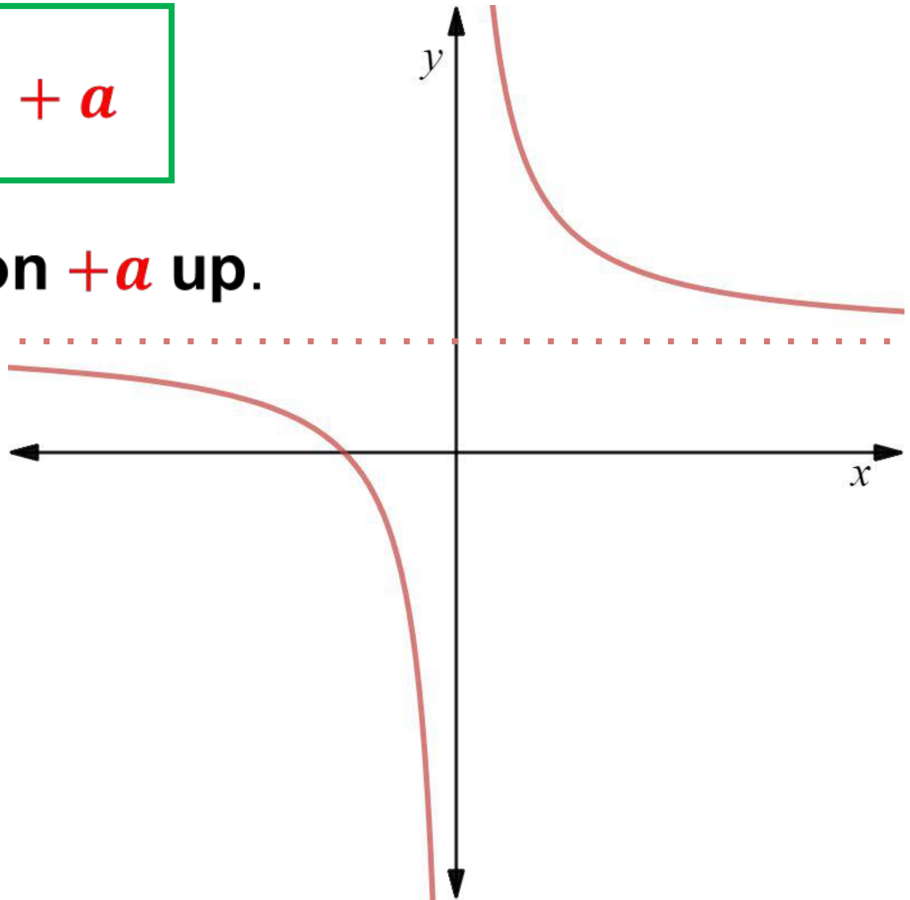
[1 Mark]

Transformations of a Curve

Graph translations of the **curve** $y = f(x)$ take the **form** of:

$$y = f(x) + a$$

Translation $+a$ **up**.

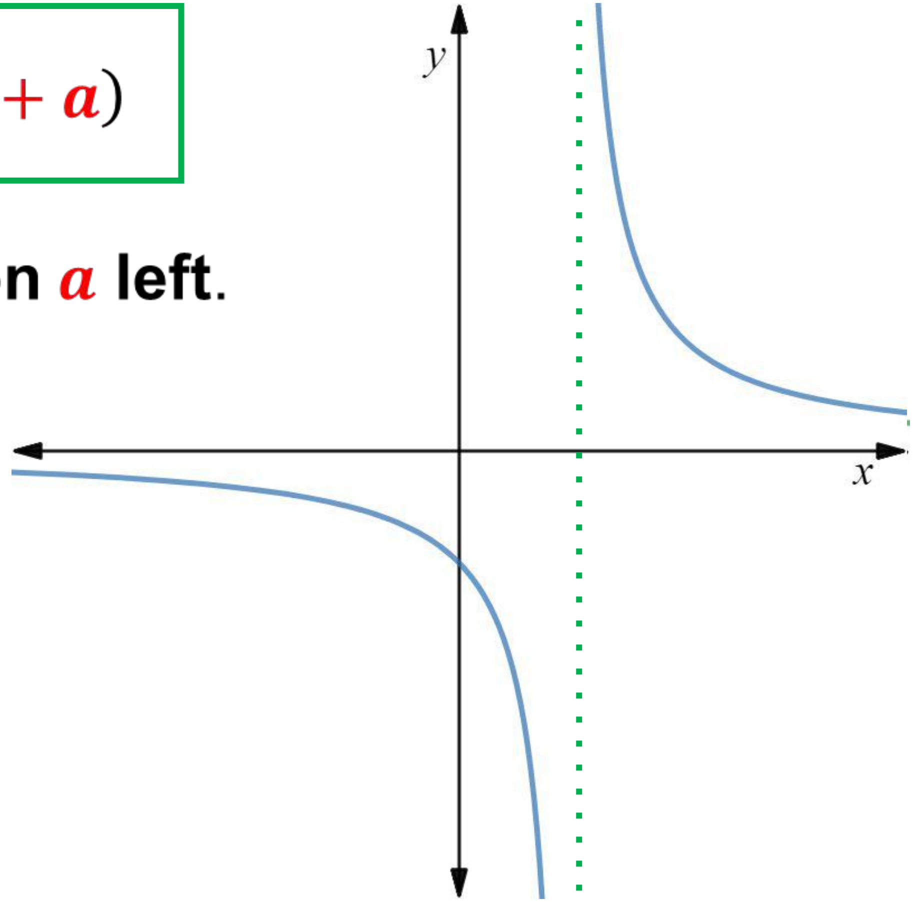


$$f(x) = \frac{1}{x}$$

Transformations of a Curve

$$y = f(x + a)$$

Translation a left.



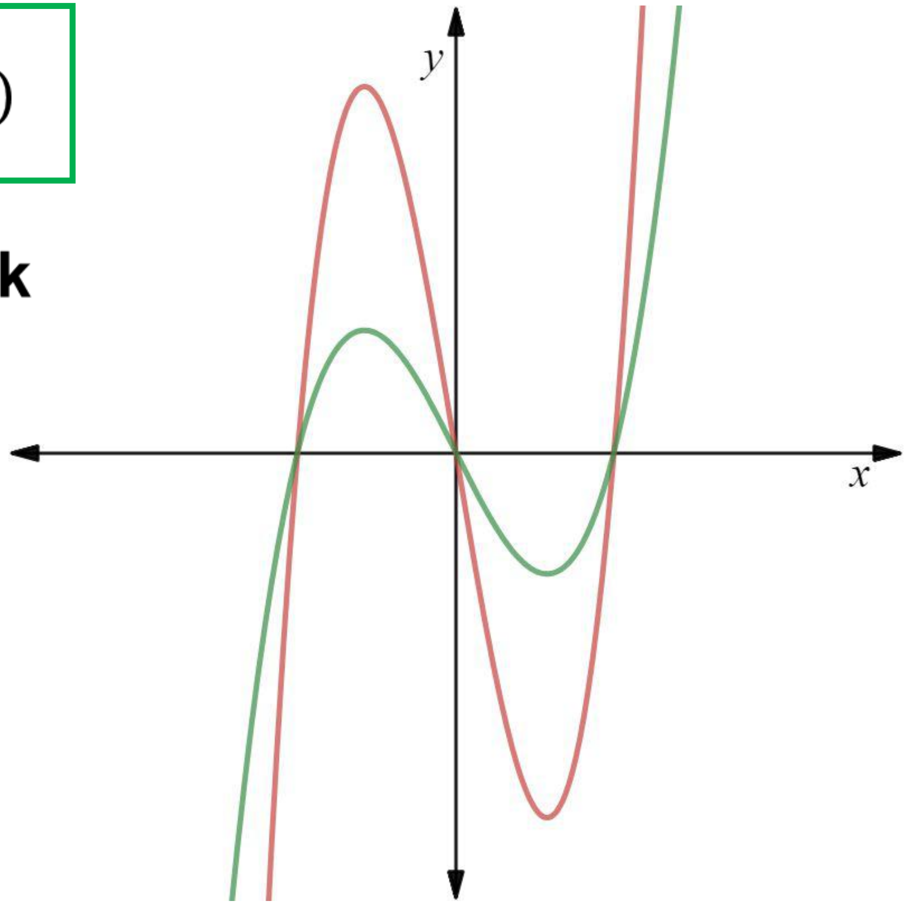
$$f(x) = \frac{1}{x}$$

Transformations of a Curve

Graph stretches of the **curve** $y = f(x)$ take the **form** of:

$$y = a \times f(x)$$

Stretch/shrink in **vertical** by **factor** a.

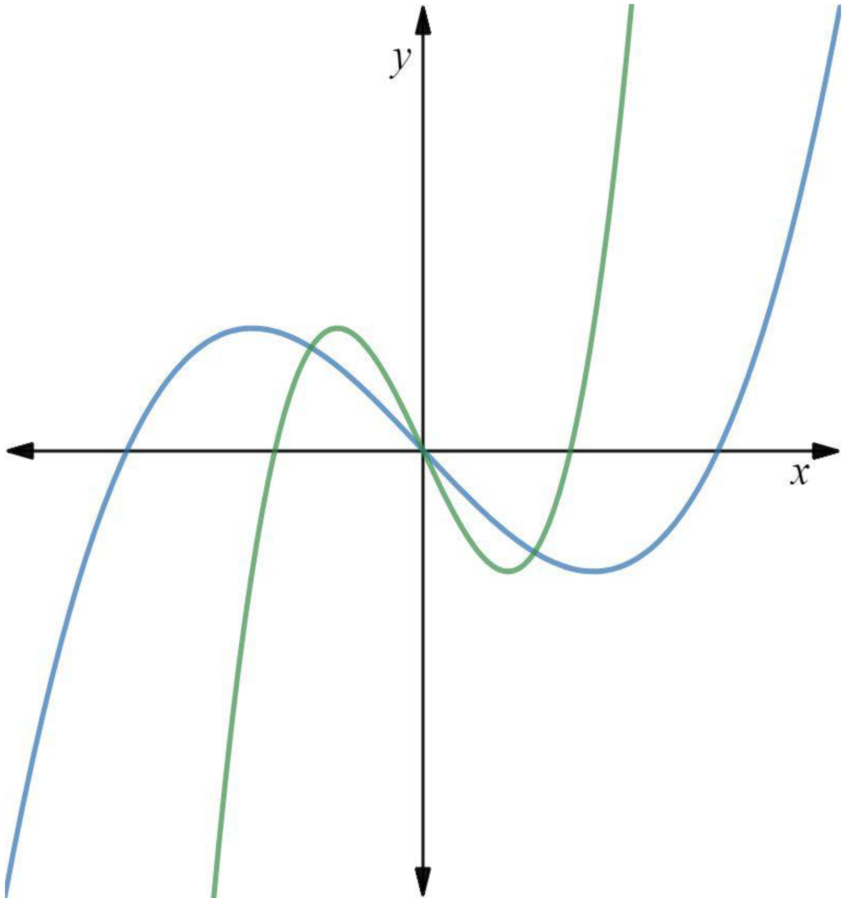


$f(x) = x^3 - 2x$

Transformations of a Curve

$$y = f(a \times x)$$

Stretch/shrink in **horizontal** by **factor** $\frac{1}{a}$.



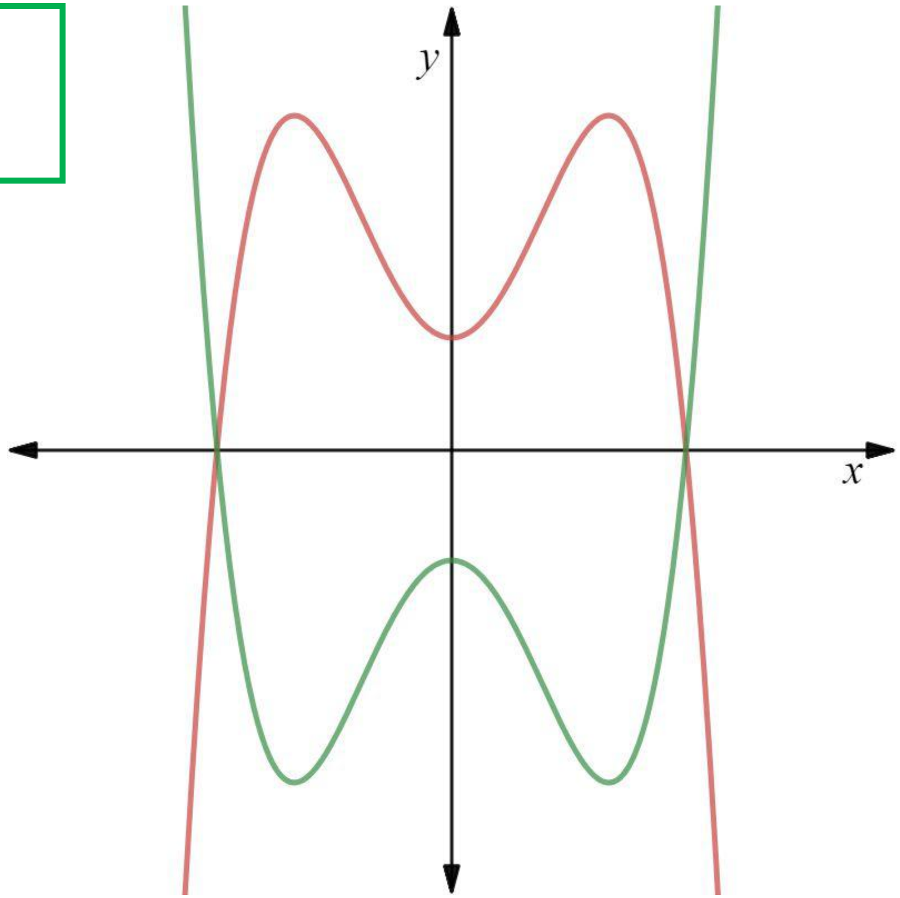
$f(x) = x^3 - 2x$

Transformations of a Curve

Graph reflections of the **curve** $y = f(x)$ take the **form** of:

$$y = -f(x)$$

Reflection in x-axis.

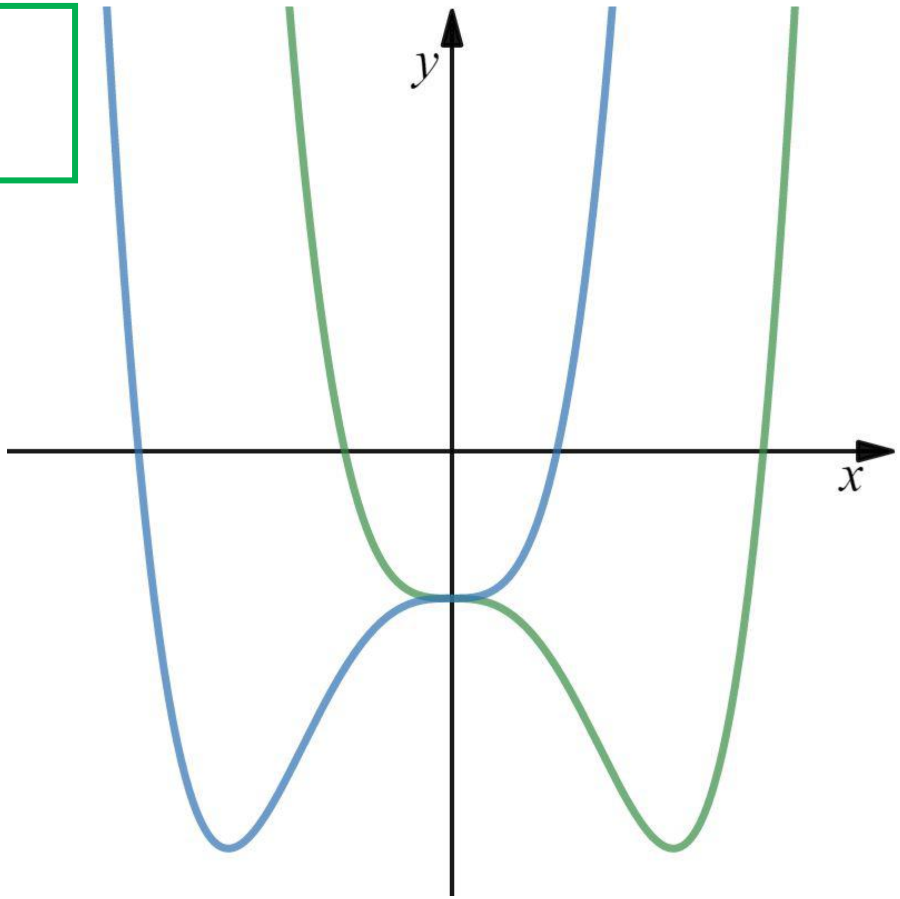


$$f(x) = (x^2 - 2)^3 - 6$$

Transformations of a Curve

$y = f(-x)$

Reflection in y-axis.



$f(x) = x^4 - 2x^3 - 1$

Looks like **graph transformations**

Exemplar Exam Question

1) The following graph plots the function $y = (x+1)^{\frac{7}{5}}$.
On the axes provided, sketch the graphs for

Complicated function, but we **don't need the details**

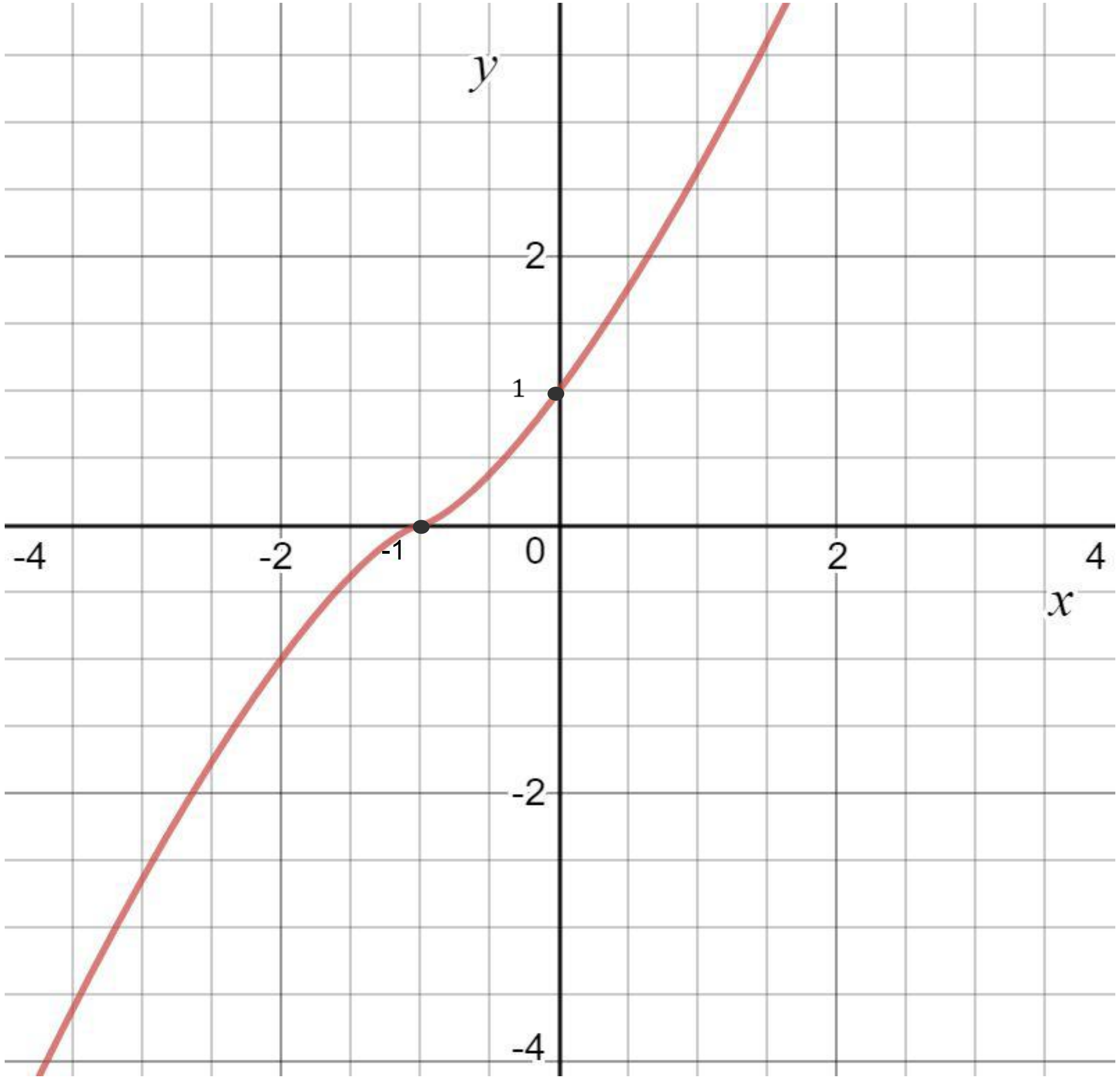
(i) $y = (x-1)^{\frac{7}{5}}$
(ii) $y = 2(x+1)^{\frac{7}{5}}$
(iii) $y = (-x+1)^{\frac{7}{5}}$

On your sketch, you should clearly state the points where the curve crosses the x and y axes

[3 marks]

Sketch implies not exact, but **we need these points to be**. **Calculate** or use knowledge of **graph transforms**

Each graph worth 1 mark.



$$y = (x+1)^{\frac{7}{5}}$$

Exemplar Exam Question Answer

Plan out what to do

Compare to what we know about graph transforms

Let $f(x) = (x+1)^{\frac{7}{5}}$

Then we're looking for

(i) $y = (x-1)^{\frac{7}{5}} = f(x-2)$

(ii) $y = 2(x+1)^{\frac{7}{5}} = 2f(x)$

(iii) $y = (-x+1)^{\frac{7}{5}} = f(-x)$

[1 Mark]

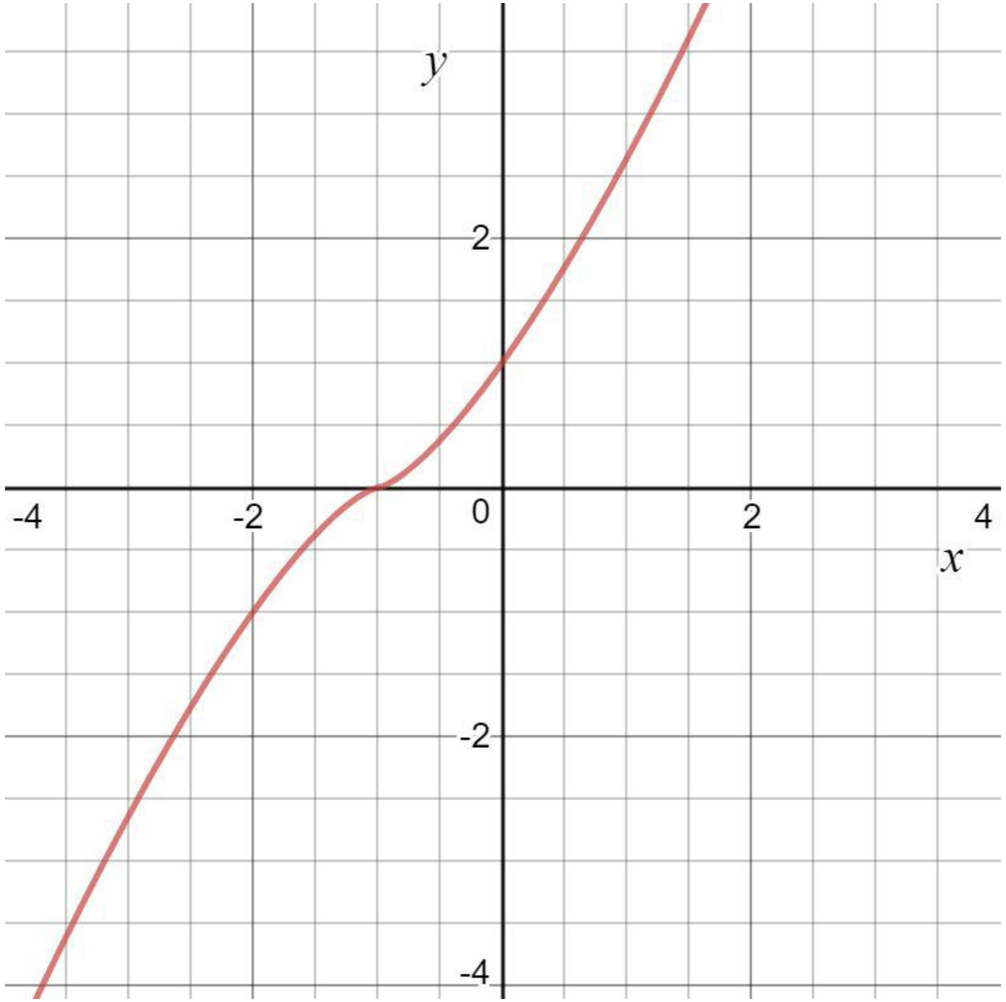
Exemplar Exam Question Answer

(i) Find points for $(x-1)^{\frac{7}{5}}$ and sketch

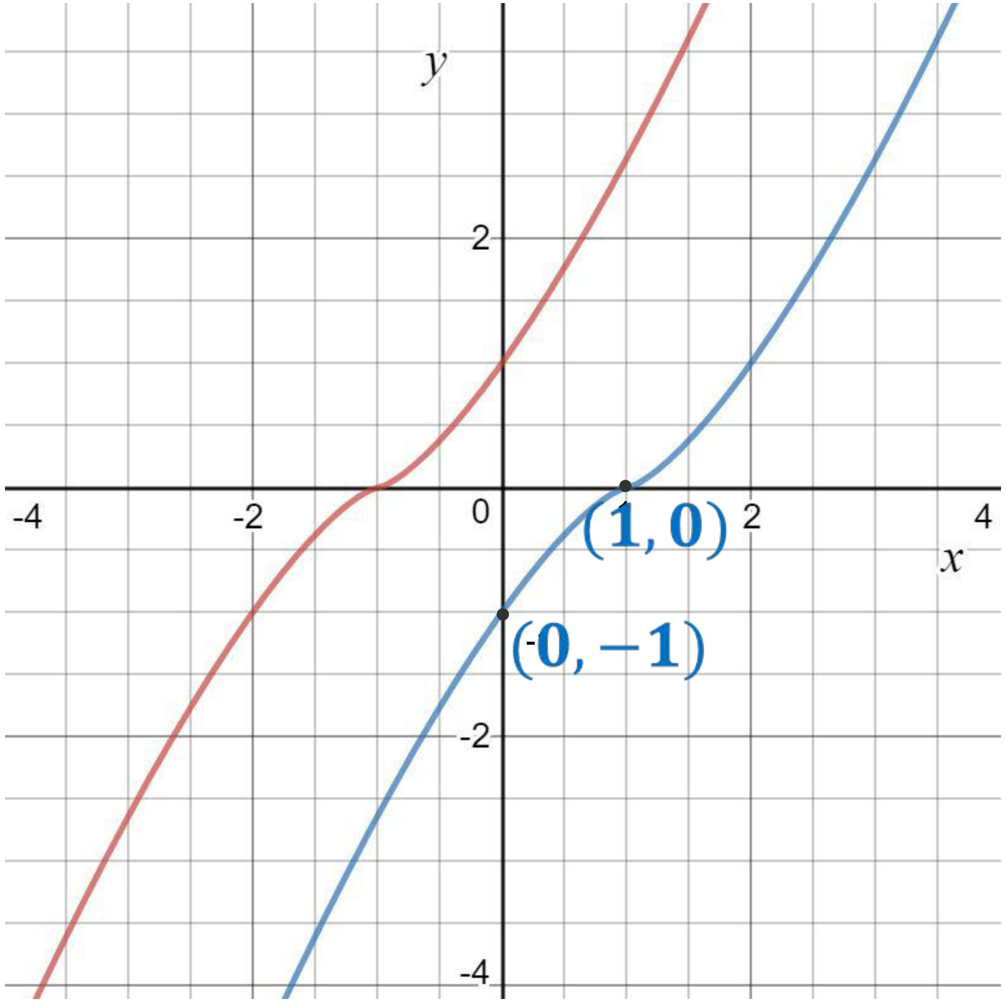
Function of form $f(x-2)$, so we translate graph right by 2.

Take known points and move 2 to right to find transformation of graph.

Exemplar Exam Question Answer



Exemplar Exam Question Answer



[1 Mark]

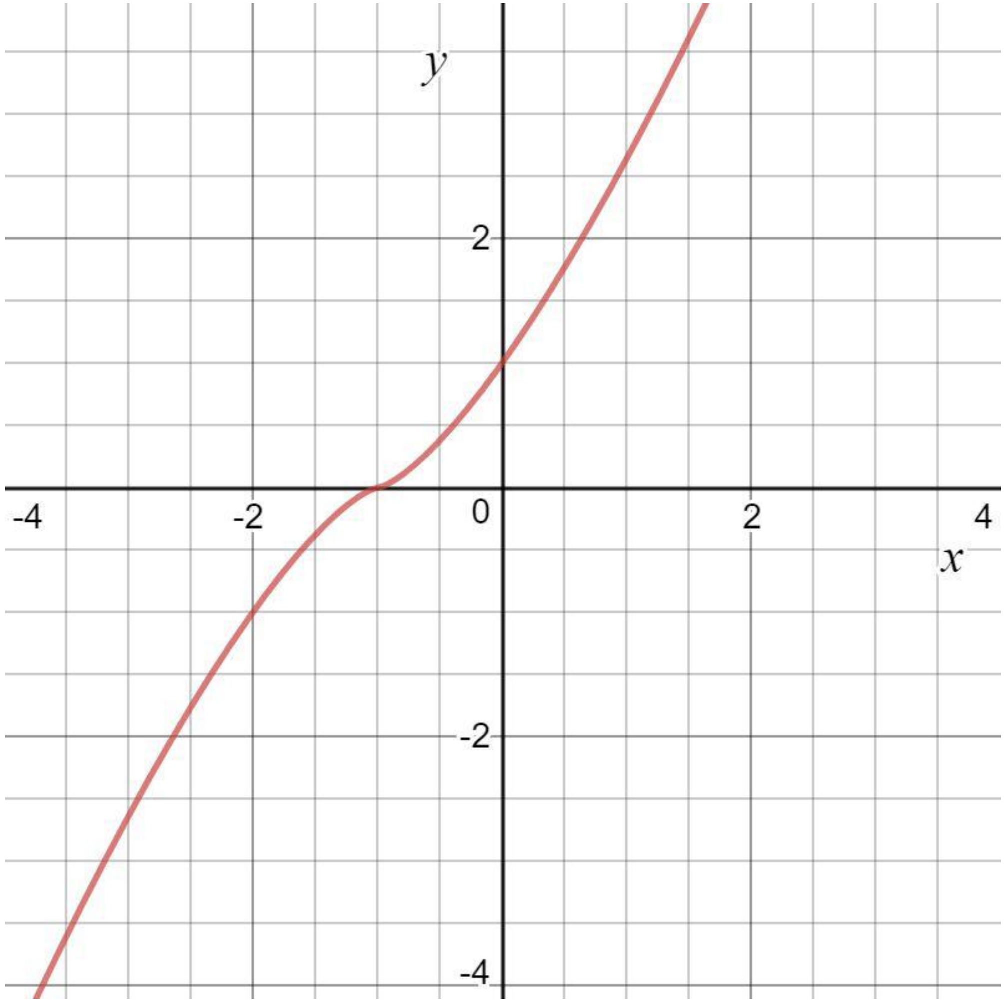
Exemplar Exam Question Answer

(ii) Find points for $2(x+1)^{\frac{7}{5}}$ and sketch

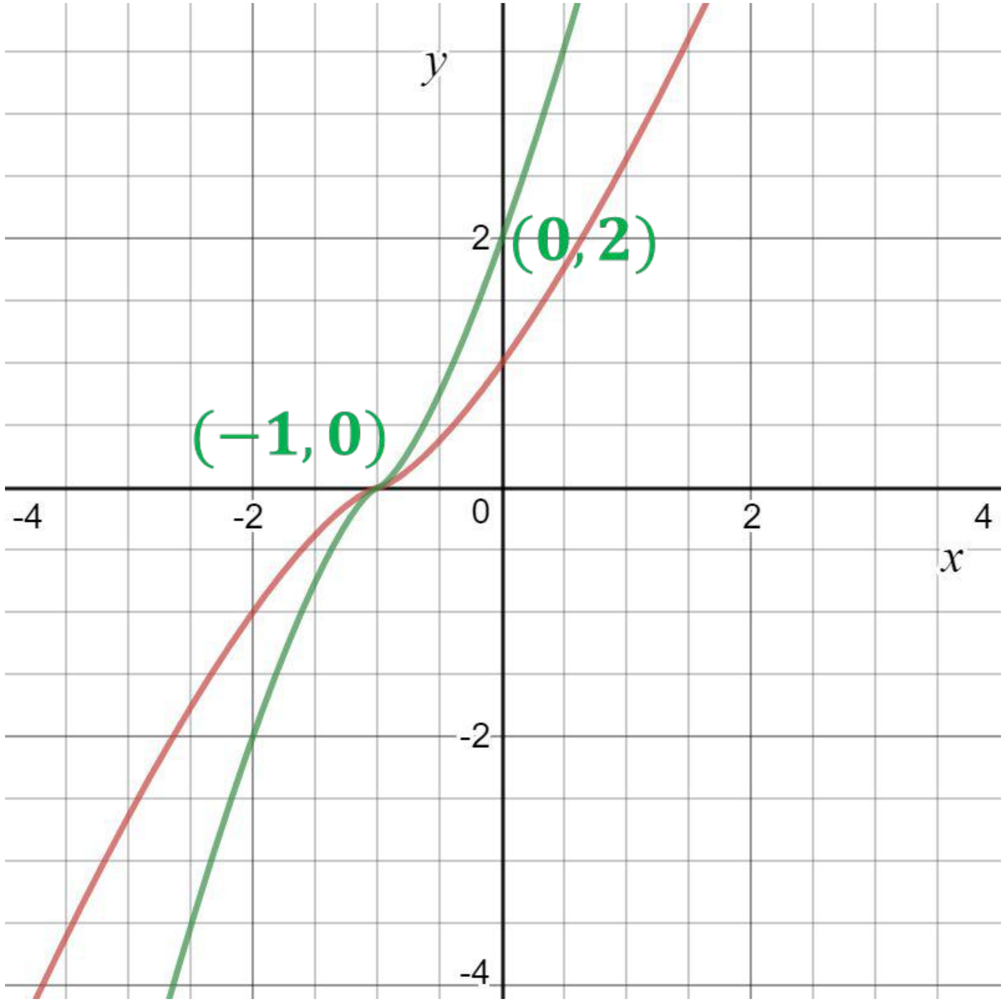
Function of form $2f(x)$ so we stretch graph vertically by 2.

Take known points and multiply their y values by 2 to find transformation of graph.

Exemplar Exam Question Answer



Exemplar Exam Question Answer



[1 Mark]

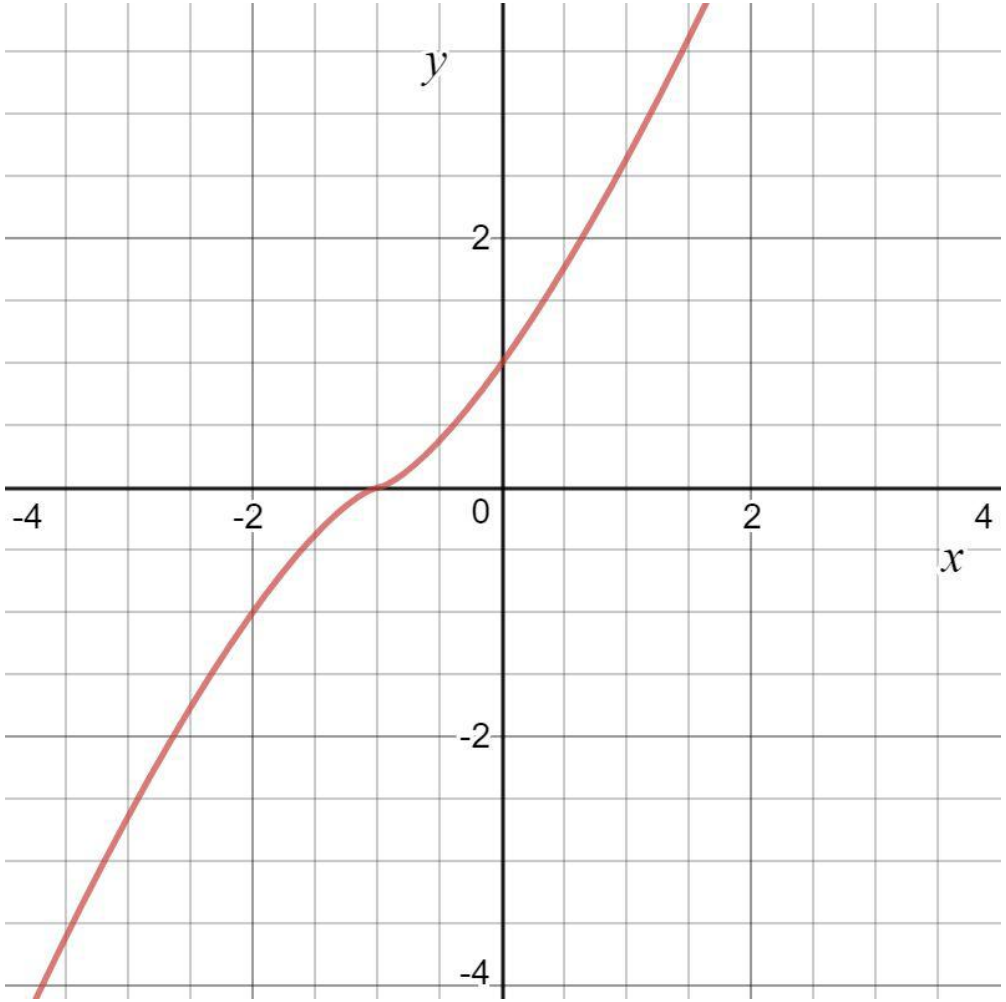
Exemplar Exam Question Answer

(iii) Find points for $(-x+1)^{\frac{7}{5}}$ and sketch

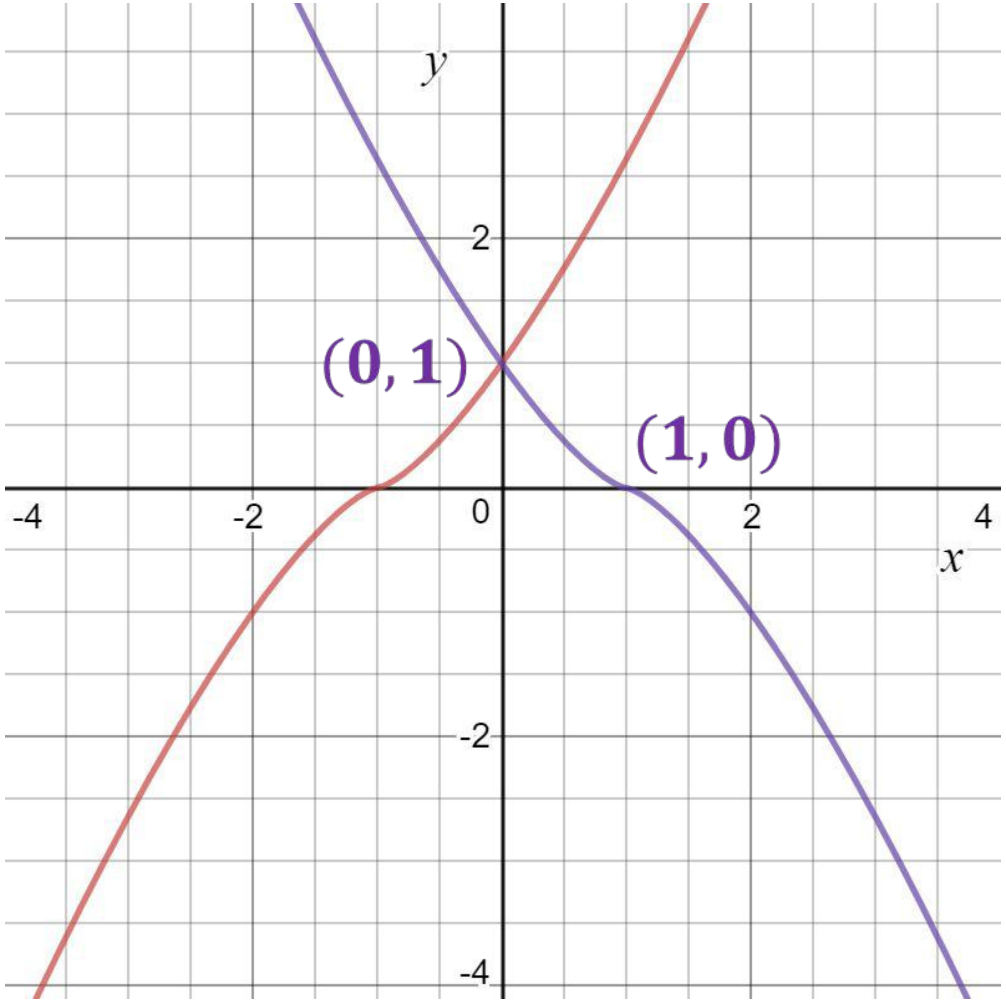
Function of form $f(-x)$ so we reflect graph in y-axis.

Take known points and reflect in y-axis to find transformation of graph.

Exemplar Exam Question Answer

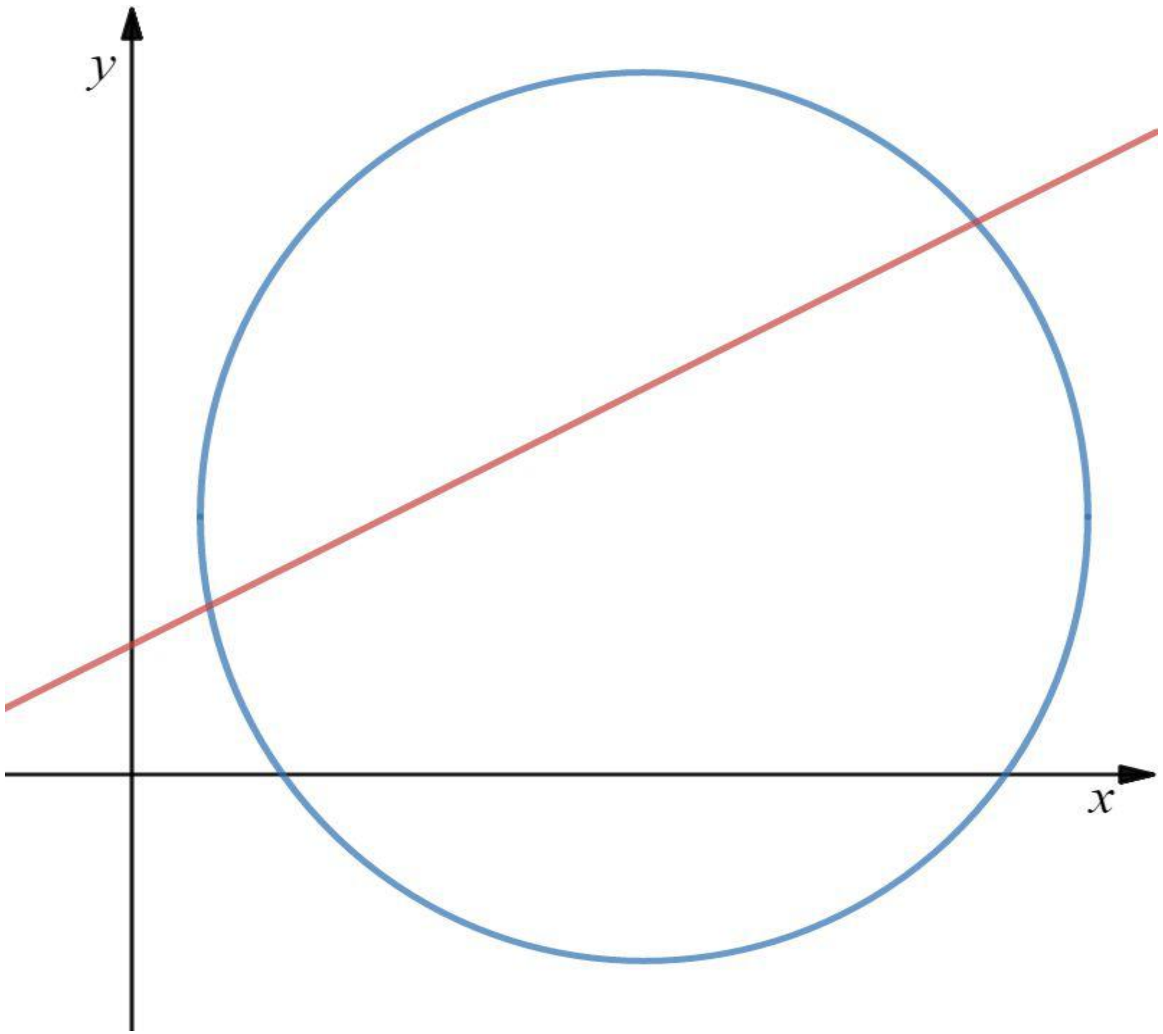


Exemplar Exam Question Answer



[1 Mark]

Coordinate Geometry



Specification Points - AQA

	Content
C1	Understand and use the equation of a straight line, including the forms $y - y_1 = m(x - x_1)$ and $ax + by + c = 0$; gradient conditions for two straight lines to be parallel or perpendicular. Be able to use straight line models in a variety of contexts.

	Content
C2	Understand and use the coordinate geometry of the circle including using the equation of a circle in the form $(x - a)^2 + (y - b)^2 = r^2$; completing the square to find the centre and radius of a circle; use of the following properties: • the angle in a semicircle is a right angle • the perpendicular from the centre to a chord bisects the chord • the radius of a circle at a given point on its circumference is perpendicular to the tangent to the circle at that point.

	Content
G1	Understand and use the derivative of $f(x)$ as the gradient of the tangent to the graph of $y = f(x)$ at a general point (x , y); the gradient of the tangent as a limit; interpretation as a rate of change; sketching the gradient function for a given curve; second derivatives; differentiation from first principles for small positive integer powers of x and for sin x and cos x Understand and use the second derivative as the rate of change of gradient; connection to convex and concave sections of curves and points of inflection.

	Content
G3	Apply differentiation to find gradients, tangents and normals, maxima and minima and stationary points, points of inflection. Identify where functions are increasing or decreasing.

Specification Points – OCR A

OCR Ref.	Subject Content	Stage 1 learners should ...
1.03 Coordinate Geometry in the x–y Plane		
1.03a	Straight lines	a) Understand and be able to use the equation of a straight line, including the forms $y = mx + c, y - y_1 = m(x - x_1)$ and $ax + by + c = 0$. *Learners should be able to draw a straight line given its equation and to form the equation given a graph of the line, the gradient and one point on the line, or at least two points on the line.* *Learners should be able to use straight lines to find:* *1. the coordinates of the midpoint of a line segment joining two points,* *2. the distance between two points and* *3. the point of intersection of two lines.*
1.03b		b) Be able to use the gradient conditions for two straight lines to be parallel or perpendicular. *i.e. For parallel lines $m_1 = m_2$ and for perpendicular lines $m_1 m_2 = -1$.*
1.03c		c) Be able to use straight line models in a variety of contexts.

1.07 Differentiation		
1.07a	Gradients	a) Understand and be able to use the derivative of $f(x)$ as the gradient of the tangent to the graph of $y = f(x)$ at a general point (x, y).
1.07b		b) Understand and be able to use the gradient of the tangent at a point where $x = a$ as: 1. the limit of the gradient of a chord as x tends to a 2. a rate of change of y with respect to x. *Learners should be able to use the notation $\frac{dy}{dx}$ to denote the rate of change of y with respect to x.* *Learners should be able to use the notations $f'(x)$ and $\frac{dy}{dx}$ and recognise their equivalence.*
1.07c		c) Understand and be able to sketch the gradient function for a given curve.
1.07d 1.07f		d) Understand and be able to find second derivatives. *Learners should be able to use the notations $f''(x)$ and $\frac{d^2y}{dx^2}$ and recognise their equivalence.*
1.07e		e) Understand and be able to use the second derivative as the rate of change of gradient. *e.g. For distinguishing between maximum and minimum*

1.03d	Circles	d) Understand and be able to use the coordinate geometry of a circle including using the equation of a circle in the form $(x - a)^2 + (y - b)^2 = r^2$. *Learners should be able to draw a circle given its equation or to form the equation given its centre and radius.*
1.03e		e) Be able to complete the square to find the centre and radius of a circle.
1.03f		f) Be able to use the following circle properties in the context of problems in coordinate geometry: 1. the angle in a semicircle is a right angle, 2. the perpendicular from the centre of a circle to a chord bisects the chord, 3. the radius of a circle at a given point on its circumference is perpendicular to the tangent to the circle at that point. *Learners should also be able to investigate whether or not a line and a circle or two circles intersect.*

f) Understand and be able to use the second derivative in connection to convex and concave sections of curves and points of inflection. *In particular, learners should know that:* *1. if $f''(x) > 0$ on an interval, the function is convex in that interval;* *2. if $f''(x) < 0$ on an interval the function is concave in that interval;* *3. if $f''(x) = 0$ and the curve changes from concave to convex or vice versa there is a point of inflection.*

1.07m 1.07p	Tangents, normals, stationary points, increasing and decreasing functions	m) Be able to apply differentiation to find the gradient at a point on a curve and the equations of tangents and normals to a curve.	p) Be able to apply differentiation to find points of inflection on a curve. *In particular, learners should know that if a curve has a point of inflection at x then $f''(x) = 0$ and there is a sign change in the second derivative on either side of x; if also $f'(x) = 0$ at that point, then the point of inflection is a stationary point, but if $f'(x) \neq 0$ at that point, then the point of inflection is not a stationary point.*
1.07n		n) Be able to apply differentiation to find and classify stationary points on a curve as either maxima or minima. *Classification may involve use of the second derivative or first derivative or other methods.*	
1.07o		o) Be able to identify where functions are increasing or decreasing. *i.e. To be able to use the sign of $\frac{dy}{dx}$ to determine whether the function is increasing or decreasing.*	

Specification Points – OCR MEI

Mg1	Know and be able to use the relationship between the gradients of parallel lines and perpendicular lines.	For parallel lines $m_1 = m_2$. For perpendicular lines $m_1 m_2 = -1$.
g2	Be able to calculate the distance between two points.	
g3	Be able to find the coordinates of the midpoint of a line segment joining two points.	
g4	Be able to form the equation of a straight line.	Including $y - y_1 = m(x - x_1)$ and $ax + by + c = 0$
g5	Be able to draw a line given its equation.	By using gradient and intercept or intercepts with axes as well as by plotting points.
g6	Be able to find the point of intersection of two lines.	By solution of simultaneous equations.
g7	Be able to use straight line models.	In a variety of contexts; includes considering the assumptions that lead to a straight line model.

g9	Be able to find the point(s) of intersection of a line and a circle.
g10	Understand and use the equation of a circle in the form $(x - a)^2 + (y - b)^2 = r^2$.
g11	Know and be able to use the following properties: • the angle in a semicircle is a right angle; • the perpendicular from the centre of a circle to a chord bisects the chord; • the radius of a circle at a given point on its circumference is perpendicular to the tangent to the circle at that point.

Mc1	Know and use that the gradient of a curve at a point is given by the gradient of the tangent at the point.	
c2	Know and use that the gradient of the tangent at a point A on a curve is given by the limit of the gradient of chord AP as P approaches A along the curve.	
c3	Understand and use the derivative of $f(x)$ as the gradient of the tangent to the graph of $y = f(x)$ at a general point (x, y). Know that the gradient function $\frac{dy}{dx}$ gives the gradient of the curve and measures the rate of change of y with respect to x.	Be able to deduce the units of rate of change for graphs modelling real situations. The term derivative of a function.
c4	Be able to sketch the gradient function for a given curve.	

c6	Understand and use the second derivative as the rate of change of gradient.	
c7	Be able to use differentiation to find stationary points on a curve: maxima and minima.	Distinguish between maximum and minimum turning points.
c8	Understand the terms increasing function and decreasing function and be able to find where the function is increasing or decreasing.	In relation to the sign of $\frac{dy}{dx}$.
c9	Be able to find the equation of the tangent and normal at a point on a curve.	

Specification Points - Edexcel

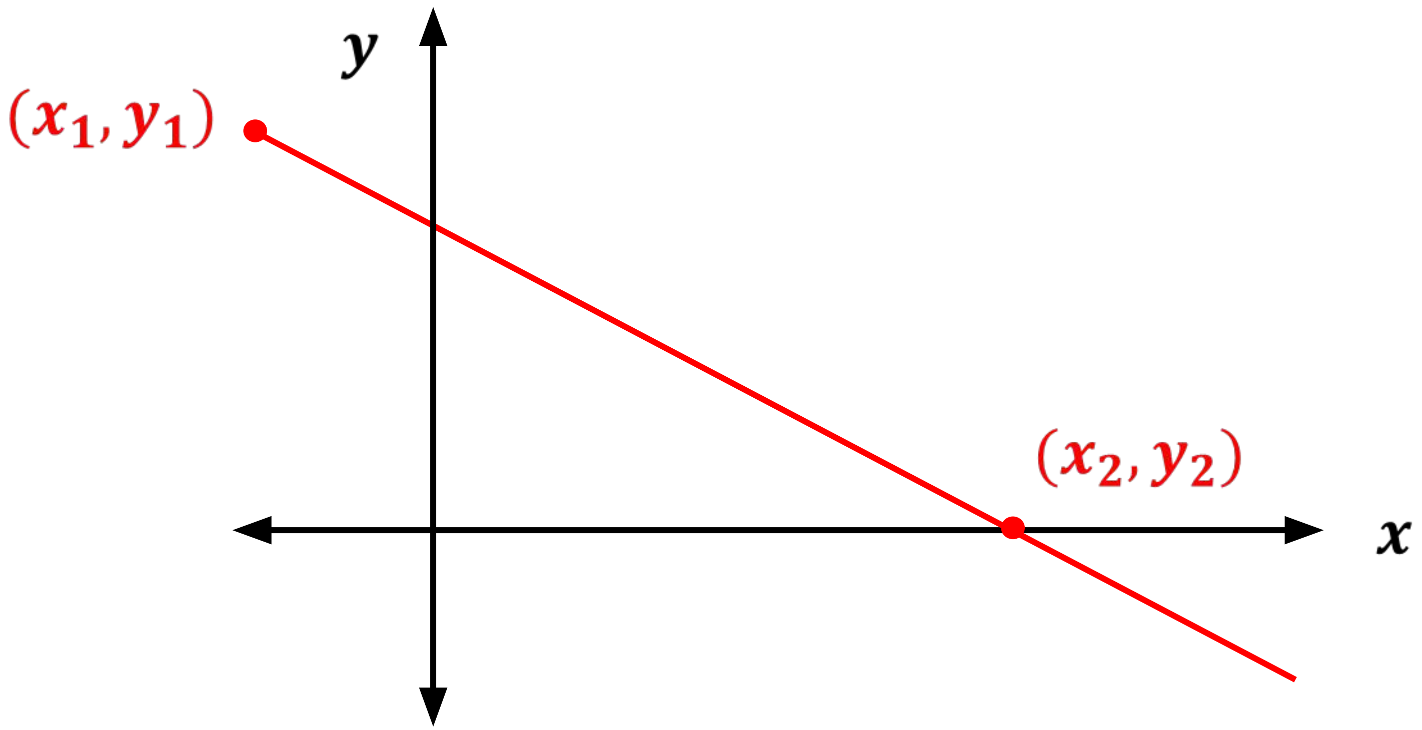
3 **Coordinate geometry in the (x,y) plane**	3.1	**Understand and use the equation of a straight line, including the forms $y - y_1 = m(x - x_1)$ and $ax + by + c = 0$;** **Gradient conditions for two straight lines to be parallel or perpendicular.** **Be able to use straight line models in a variety of contexts.**	**To include the equation of a line through two given points, and the equation of a line parallel (or perpendicular) to a given line through a given point.** **$m' = m$ for parallel lines and $m' = -\frac{1}{m}$ for perpendicular lines** **For example, the line for converting degrees Celsius to degrees Fahrenheit, distance against time for constant speed, etc.**
	3.2	**Understand and use the coordinate geometry of the circle including using the equation of a circle in the form $(x - a)^2 + (y - b)^2 = r^2$** **Completing the square to find the centre and radius of a circle; use of the following properties:** • **the angle in a semicircle is a right angle** • **the perpendicular from the centre to a chord bisects the chord** • **the radius of a circle at a given point on its circumference is perpendicular to the tangent to the circle at that point.**	**Students should be able to find the radius and the coordinates of the centre of the circle given the equation of the circle, and vice versa.** **Students should also be familiar with the equation $x^2 + y^2 + 2fx + 2gy + c = 0$** **Students should be able to find the equation of a circumcircle of a triangle with given vertices using these properties.** **Students should be able to find the equation of a tangent at a specified point, using the perpendicular property of tangent and radius.**

7 **Differentiation**	7.1	**Understand and use the derivative of $f(x)$ as the gradient of the tangent to the graph of $y = f(x)$ at a general point (x, y); the gradient of the tangent as a limit; interpretation as a rate of change** **sketching the gradient function for a given curve** **second derivatives**	**Know that $\frac{dy}{dx}$ is the rate of change of y with respect to x.** **The notation $f'(x)$ may be used for the first derivative and $f''(x)$ may be used for the second derivative.** **Given for example the graph of $y = f(x)$, sketch the graph of $y = f'(x)$ using given axes and scale. This could relate speed and acceleration for example.**
7 **Differentiation** *continued*	7.1 *cont.*	**Understand and use the second derivative as the rate of change of gradient;** connection to convex and concave sections of curves and points of inflection.	Use the condition $f''(x) > 0$ implies a minimum and $f''(x) < 0$ implies a maximum for points where $f'(x) = 0$ Know that at an inflection point $f''(x)$ changes sign. Consider cases where $f''(x) = 0$ and $f'(x) = 0$ where the point may be a minimum, a maximum or a point of inflection (e.g. $y = x^n$, $n > 2$)

Equation of a Straight Line

A **straight line** has an **equation** which can be **rearranged** to the **form**:

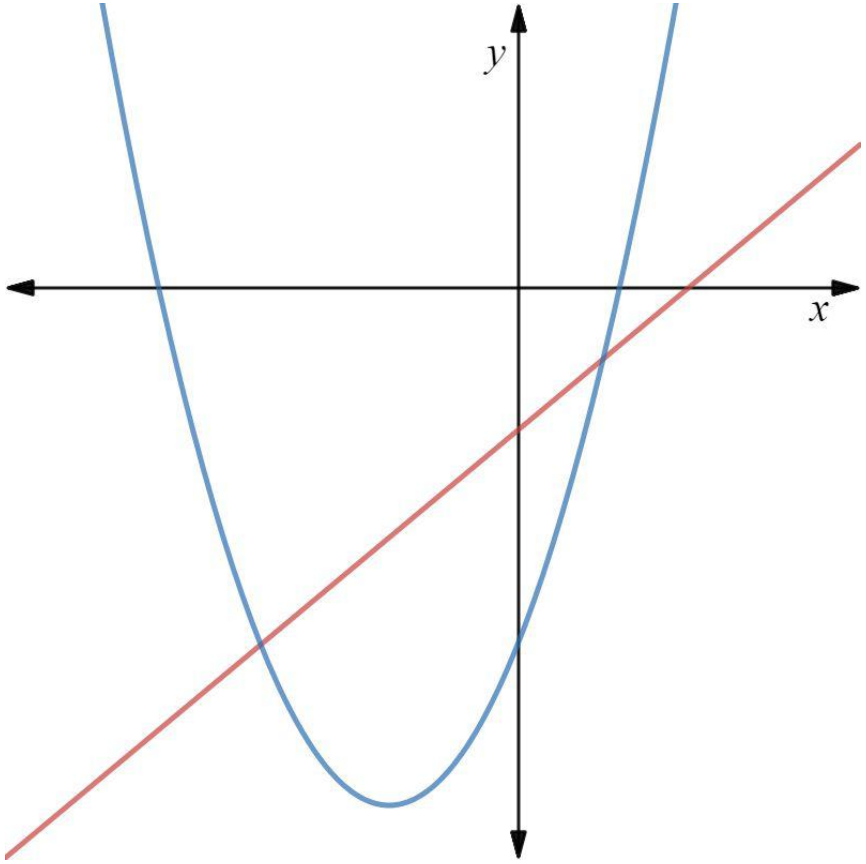
$$y = mx + c$$



Equation of a Straight Line

The **points of intersection** between a **straight line** and a **curve** can be found by **equating** their **equations**:

$y = x - 1$ $\quad\quad$ $y = x^2 + 3x - 6$



The **discriminant** of this **equation** shows how many times the **graphs intersect**.

$\Delta = b^2 - 4ac$

Exemplar Exam Question

Straight line graphs

Can find a **straight line** with this information

1) A straight line passes though the points $(2, 3)$ and $(0, -1)$.

(i) Derive the equation for the line, giving your answer in the form $y = mx + c$

(ii) The line is tangent to a curve with equation $y = kx^2 + 8x + 2$. Calculate the value of k

[4 Marks]

Don't need to find **actual tangent point**, use **discriminants** to see if one exists

4 stages across questions, **likely one for (i)** (for finding m **and** c), and **three stages to calculating** k

Exemplar Exam Question Answer

$y = mx + c$ $(2, 3), (0, -1)$

(i) Calculate value of c

Remember c is y-intercept of graph

That is the y-value when $x = 0$

This has been given in the question. Can read the value for c off the second co-ordinate

$c = -1$

Exemplar Exam Question Answer

$y = mx - 1$ $(2, 3), (0, -1)$

Calculate value of m

m is the gradient of the line

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Substitute in coordinate values

$$m = \frac{3-(-1)}{2-0} = \frac{4}{2} = 2$$

Therefore $y = 2x - 1$ **[1 Mark]**

Exemplar Exam Question Answer

$$y = 2x - 1 \qquad\qquad y = kx^2 + 8x + 2$$

(ii) Equate both equations

Tangent is a type of intersection. Lines intersect when equations are equal

$$2x - 1 = kx^2 + 8x + 2$$

Bring all terms to one side

$$0 = kx^2 + 6x + 3$$

[1 Mark]

Exemplar Exam Question Answer

$$kx^2 + 6x + 3 = 0$$

Derive discriminant

For quadratic equation $ax^2 + bx + c = 0$, discriminant is given by
$\Delta = b^2 - 4ac$

Use formula on our quadratic

$\Delta = 6^2 - (4 \times 3)k$

$= 36 - 12k$

[1 Mark]

Exemplar Exam Question Answer

$$\Delta = 36 - 12k$$

Derive k

Line is tangent when discriminant is zero

Let $\Delta = 0$

$36 - 12k = 0$

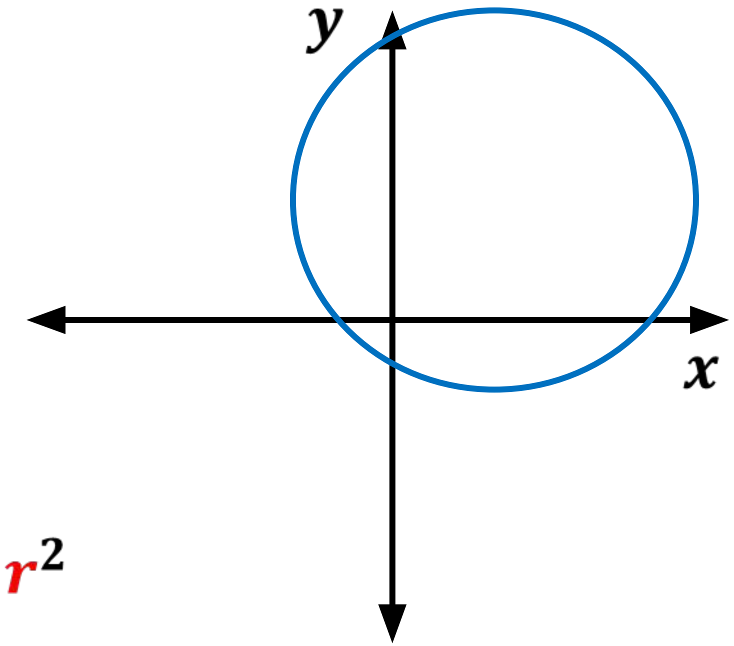
Solve for k

$12k = 36 \quad \Rightarrow k = 3$ **[1 Mark]**

The Coordinate Geometry of a Circle

A **circle** of **radius** r with its **centre** at the **point** (a, b) is represented by the **general equation**:

$$(x-a)^2+(y-b)^2=r^2$$



- An **equation** of the **form** $x^2+px+y^2+qy+c=0$ can be **converted** to the **general equation** for a **circle** by **completing the square** for both x and y.

$$x^2+px+y^2+qy+c=0 \longrightarrow (x-a)^2+(y-b)^2=r^2$$

Completing the Square:

$$x^2+px=\left(x+\frac{p}{2}\right)^2-\frac{p^2}{4}$$

$$y^2+qy=\left(y+\frac{q}{2}\right)^2-\frac{q^2}{4}$$

Exemplar Exam Question

Clearly a circle, but will need to **rearrange to correct form**

1) A circle has the equation $x^2 - 4x + y^2 + 6y - 5 = 0$

(i) Find the co-ordinates of the centre of the circle.

(ii) Find the radius of the circle.

(ii) Verify if the point $(-1, 0)$ lies on the circle.

[4 Marks]

Three parts, **two key features of circle to find** along with a **basic application**

5 stages across three parts. **Two marks for one** and **one each for others**

Exemplar Exam Question Answer

$$x^2 - 4x + y^2 + 6y - 5 = 0$$

(i) Complete the square for x and y

$$(x-2)^2 - 4 + y^2 + 6y - 5 = 0$$

$$(x-2)^2 - 4 + (y+3)^2 - 9 - 5 = 0$$

Add up constants and move to other side

$$(x-2)^2 + (y+3)^2 = 18$$

[1 Mark]

Exemplar Exam Question Answer

$$(x-2)^2+(y-(-3))^2=18$$

Read off circle's centre

Circle with equation $(x-a)^2+(y-b)^2=r^2$ has centre (a, b)

Apply to equation to find centre

Circle has centre $(2, -3)$

[1 Mark]

Exemplar Exam Question Answer

$$(x-2)^2+(y+3)^2 = 18$$

(ii) Read off circle's radius

Circle with equation $(x-a)^2+(y-b)^2 = r^2$ has radius r

Apply to equation to find radius

Circle has radius $\sqrt{18} = 3\sqrt{2}$

[1 Mark]

Exemplar Exam Question Answer

$(x-2)^2+(y+3)^2=18$ $(-1, 0)$

(iii) Substitute values for x and y into equation

Trying to find if $(-1, 0)$ is on circle

Substitute $x=-1$ and $y=0$ into equation

$$(-1-2)^2+(0+3)^2=18$$

$$(-3)^2+(3^2)=18$$

$$9+9=18$$

So point <u>does</u> lie on circle **[1 Mark]**

MINI MOCK PAPER

Exam Question

1. A curve C has the equation $y = x^3 + x^2 + x + 1$.

(i) The line l_1, given by $y = 2x - 1$ intersects with C at the point $(-2, -5)$. Show, without use of a graph, that l_1 and C don't meet at any other points.

[5 marks]

(ii) A second line l_2 is tangent to C at the point $(1, 4)$. Derive an equation for l_2 in the form $y = mx + c$

[4 marks]

Exam Question Answer

(i) Equate C and l_1

Lines intersect when equations are equal.

$$x^3 + x^2 + x + 1 = 2x - 1$$

$$x^3 + x^2 - x + 2 = 0$$

[1 Mark]

Factorise expression to quadratic

Let $f(x) = x^3 + x^2 - x + 2$

From question, we know that $f(x) = 0$ when $x = -2$, i.e $f(-2) = 0$

So, by factor theorem, $(x + 2)$ is a factor of $f(x)$

Exam Question Answer

Factorise using polynomial long division

$$
\begin{array}{r l}
 & x^2 - x + 1 \\
(x+2) & \overline{)\, x^3 + x^2 - x + 2} \\
 & \underline{x^3 + 2x^2} \\
 & -x^2 - x \\
 & \underline{-x^2 - 2x} \\
 & x + 2 \\
 & \underline{x + 2} \\
 & 0
\end{array}
$$

[2 Marks]

Exam Question Answer

Calculate discriminant and draw conclusion

For quadratic $ax^2 + bx + c = 0$, discriminant given by $\Delta = b^2 - 4ac$

$$\Delta = (-1)^2 - (4 \times 1 \times 1)$$

$$= 1 - 4 = -3$$

[1 Mark]

So $\Delta < 0$

Hence $x^2 - x + 1$ has no roots

So $x = -2$ is the only root of $f(x)$, and the only point when C and l_1 intersect

[1 Mark]

Exam Question Answer

(ii) Calculate m

m is equivalent to gradient of line

Line is tangent to C, so m is equal to gradient of C at intersection point

[1 Mark]

Differentiate equation for C

$$y = x^3 + x^2 + x + 1$$

$$\frac{dy}{dx} = 3x^2 + 2x + 1$$

[1 Mark]

Exam Question Answer

Substitue in value for x

m and C intersect when $x = 1$

Substitute $x = 1$ into equation for first derivative.

$$\frac{dy}{dx} = 3x^2 + 2x + 1$$

$$\frac{dy}{dx} = 3 + 2 + 1 = 6$$

So $m = 6$

[1 Mark]

Exam Question Answer

Calculate value for C

Know l_2 passes through point $(1,4)$

Substitute $x = 1, y = 4$ into equation for l_2

$$y = 6x + c$$

$$4 = 6 + c$$

$$c = -2$$

[1 Mark]