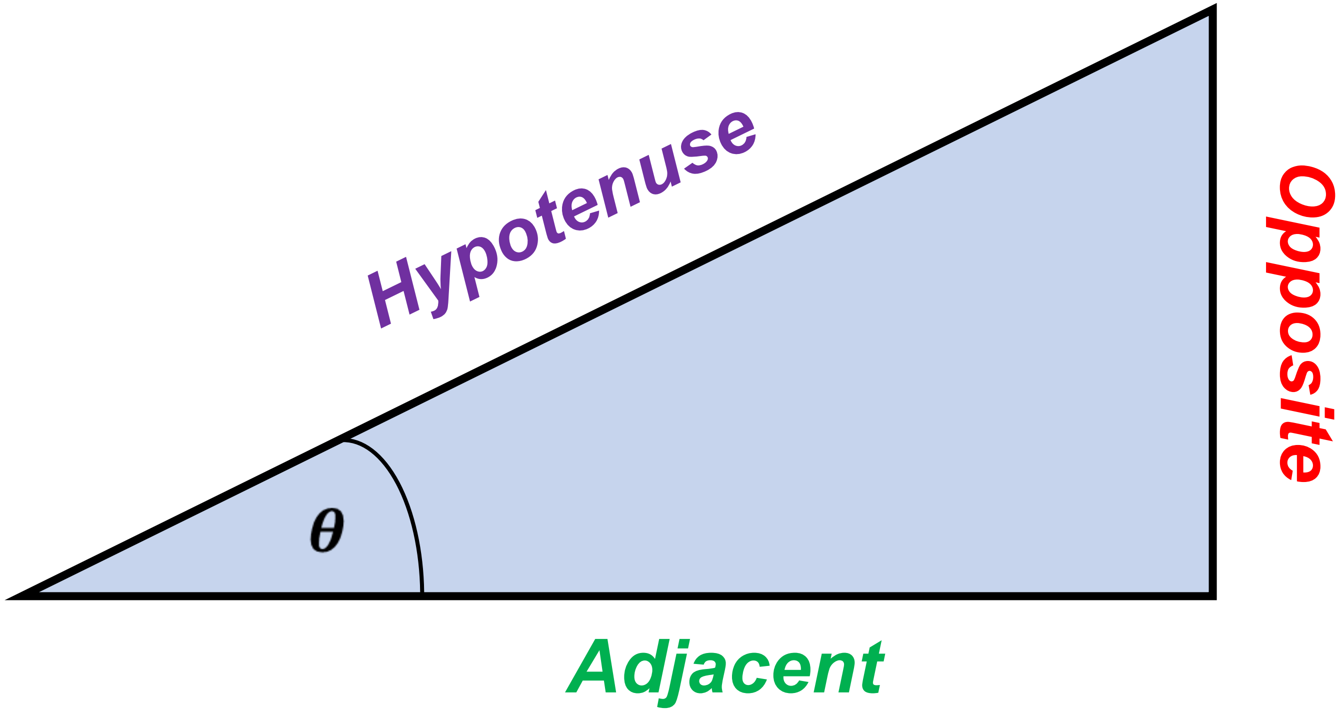
Pure Mathematics: Trigonometry



Material Covered

Basic Trigonometric Functions

1. Trigonometric Graphs.

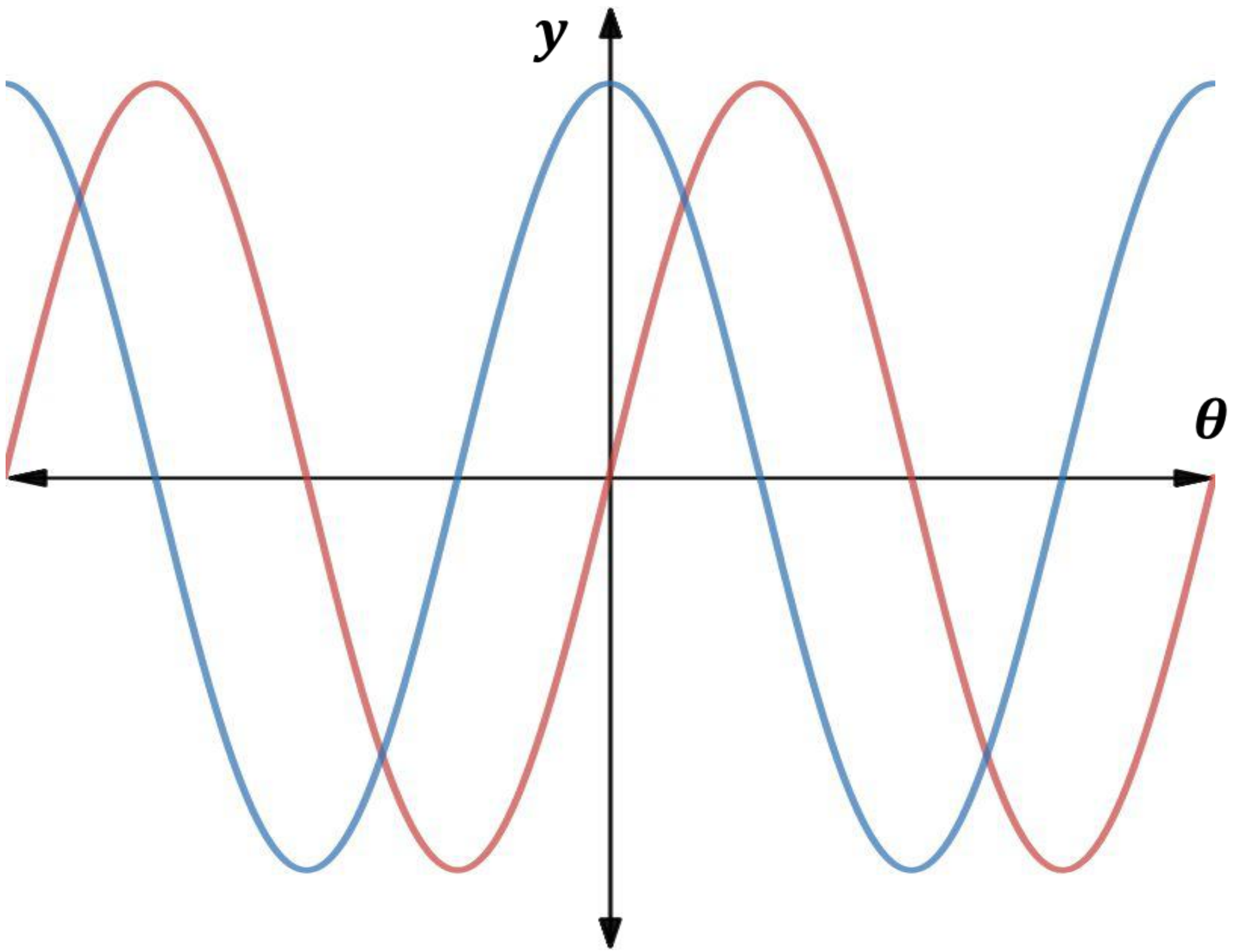
Advanced Trigonometric Functions

1. Trigonometric Equations
2. Reciprocal Trigonometric Functions.

Compound Angles

1. Compound Angle Formulae.
2. Rearranging Trigonometric Expressions.

Basic Trigonometric Functions



Specification Points - AQA

	Content
E1	Understand and use the definitions of sine, cosine and tangent for all arguments; the sine and cosine rules; the area of a triangle in the form $\frac{1}{2}ab\sin C$ Work with radian measure, including use for arc length and area of sector.

	Content
E2	Understand and use the standard small angle approximations of sine, cosine and tangent $\sin\theta \approx \theta,\ \cos\theta \approx 1 - \frac{\theta^2}{2},\ \tan\theta \approx \theta$ where θ is in radians.

	Content
E3	Understand and use the sine, cosine and tangent functions; their graphs, symmetries and periodicity. Know and use exact values of sin and cos for $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \pi$ and multiples thereof, and exact values of tan for $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \pi$ and multiples thereof.

Specification Points – OCR A

1.05a 1.05d 1.05b 1.05c	sin, cos and tan for all arguments Sine and cosine rules Radians	a) Understand and be able to use the definitions of sine, cosine and tangent for all arguments. b) Understand and be able to use the sine and cosine rules. *Questions may include the use of bearings and require the use of the ambiguous case of the sine rule.* c) Understand and be able to use the area of a triangle in the form $\frac{1}{2}ab\sin C$.	d) Be able to work with radian measure, including use for arc length and area of sector. *Learners should know the formulae $s = r\theta$ and $A = \frac{1}{2}r^2\theta$.* *Learners should be able to use the relationship between degrees and radians.*
1.05e	Small angle approximations		e) Understand and be able to use the standard small angle approximations of sine, cosine and tangent: 1. $\sin\theta \approx \theta$, 2. $\cos\theta \approx 1 - \frac{1}{2}\theta^2$, 3. $\tan\theta \approx \theta$, where θ is in radians. *e.g. Find an approximate expression for $\frac{\sin 3\theta}{1+\cos\theta}$ if θ is small enough to neglect terms in θ^3 or above.*
1.05f 1.05g	Graphs of the basic trigonometric functions Exact values of trigonometric functions	f) Understand and be able to use the sine, cosine and tangent functions, their graphs, symmetries and periodicities. *Includes knowing and being able to use exact values of $\sin\theta$ and $\cos\theta$ for $\theta = 0°, 30°, 45°, 60°, 90°, 180°$ and multiples thereof and exact values of $\tan\theta$ for $\theta = 0°, 30°, 45°, 60°, 180°$ and multiples thereof.*	g) Know and be able to use exact values of $\sin\theta$ and $\cos\theta$ for $\theta = 0, \frac{1}{6}\pi, \frac{1}{4}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi, \pi$ and multiples thereof, and exact values of $\tan\theta$ for $\theta = 0, \frac{1}{6}\pi, \frac{1}{4}\pi, \frac{1}{3}\pi, \pi$ and multiples thereof.

Specification Points – OCR MEI

Mt1	Be able to use the definitions of $\sin\theta$, $\cos\theta$ and $\tan\theta$ for any angle.	By reference to the unit circle, $\sin\theta = y,\ \cos\theta = x,\ \tan\theta = \frac{y}{x}$.
t2	Know and use the graphs of $\sin\theta$, $\cos\theta$ and $\tan\theta$ for all values of θ, their symmetries and periodicities.	Stretches, translations and reflections of these graphs. Combinations of these transformations.
*	Know and be able to use the exact values of $\sin\theta$ and $\cos\theta$ for $\theta = 0°, 30°, 45°, 60°$ and $90°$ and the exact values of $\tan\theta$ for $\theta = 0°, 30°, 45°$ and $60°$.	
t5	Understand and be able to use $\tan\theta = \frac{\sin\theta}{\cos\theta}$.	e.g. solve $\sin\theta = 3\cos\theta$ for $0° \le \theta \le 360°$.
t6	Understand and be able to use the identity $\sin^2\theta + \cos^2\theta = 1$.	e.g. solve $\sin^2\theta = \cos\theta$ for $0° \le \theta \le 360°$.

Mt8	Know and be able to use exact values of $\sin\theta$, $\cos\theta$, $\tan\theta$ for $\theta = 0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \pi$ and multiples thereof and $\sin\theta$, $\cos\theta$ for $\theta = \frac{\pi}{2}$ and multiples thereof.	
t9	Understand and use the definitions of the functions arcsin, arccos and arctan, their relationship to sin, cos and tan, their graphs and their ranges and domains.	
t10	Understand and use the definition of a radian and be able to convert between radians and degrees.	
t11	Know and be able to find the arc length and area of a sector of a circle, when the angle is given in radians.	The results $s = r\theta$ and $A = \frac{1}{2}r^2\theta$ where θ is measured in radians.
t12	Understand and use the standard small angle approximations of sine, cosine and tangent.	$\sin\theta \approx \theta,\ \cos\theta \approx 1 - \frac{\theta^2}{2}, \tan\theta \approx \theta$ where θ is in radians.

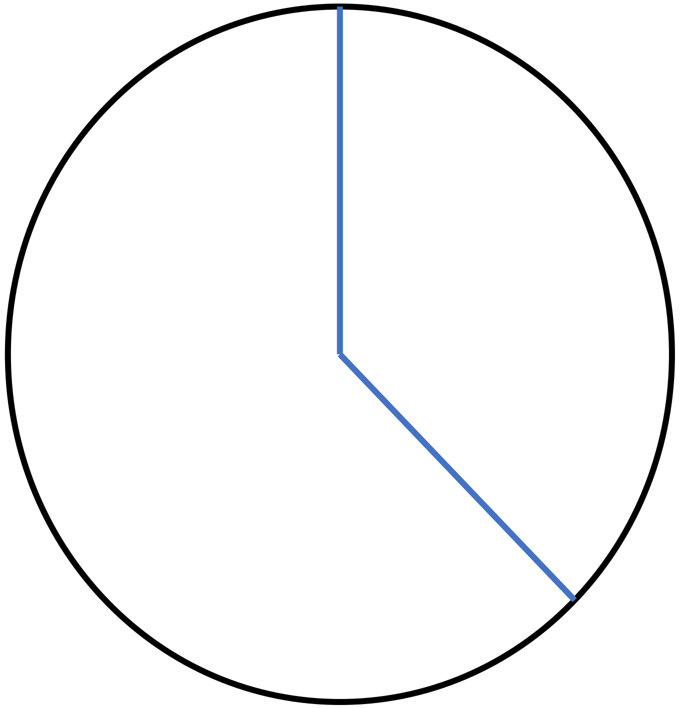
Specification Points - Edexcel

5.1	**Understand and use the definitions of sine, cosine and tangent for all arguments;** **the sine and cosine rules;** **the area of a triangle in the form $\frac{1}{2}ab\sin C$** Work with radian measure, including use for arc length and area of sector.	**Use of x and y coordinates of points on the unit circle to give cosine and sine respectively,** **including the ambiguous case of the sine rule.** Use of the formulae $s = r\theta$ and $A = \frac{1}{2}r^2\theta$ for arc lengths and areas of sectors of a circle.
5.2	Understand and use the standard small angle approximations of sine, cosine and tangent $\sin\theta \approx \theta$, $\cos\theta \approx 1 - \frac{\theta^2}{2}$, $\tan\theta \approx \theta$ Where θ is in radians.	Students should be able to approximate, e.g. $\frac{\cos 3x - 1}{x\sin 4x}$ when x is small, to $-\frac{9}{8}$

5.3	**Understand and use the sine, cosine and tangent functions; their graphs, symmetries and periodicity.** Know and use exact values of sin and cos for $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \pi$ and multiples thereof, and exact values of tan for $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \pi$ and multiples thereof.	**Knowledge of graphs of curves with equations such as $y = \sin x$, $y = \cos(x + 30°)$, $y = \tan 2x$ is expected.**

Radians

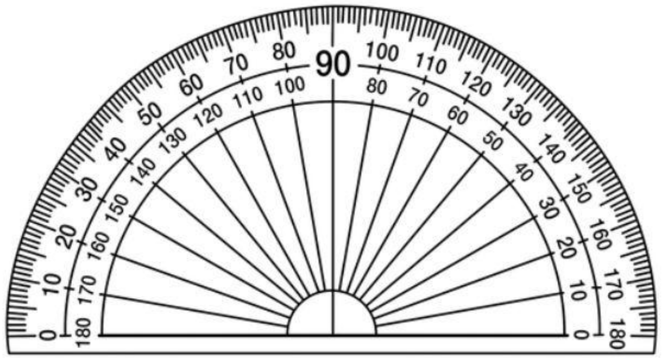
One radian (**rad**) is the **angle** formed by a **circular arc** at the **centre of a circle** when the **arc length** is **equal** to its **radius**.



$$s = r \Rightarrow \theta = 1 \text{ rad}$$

$$s = C \Rightarrow \theta = 2\pi \text{ rad}$$

$$1 \text{ rad} = \frac{180°}{\pi}$$

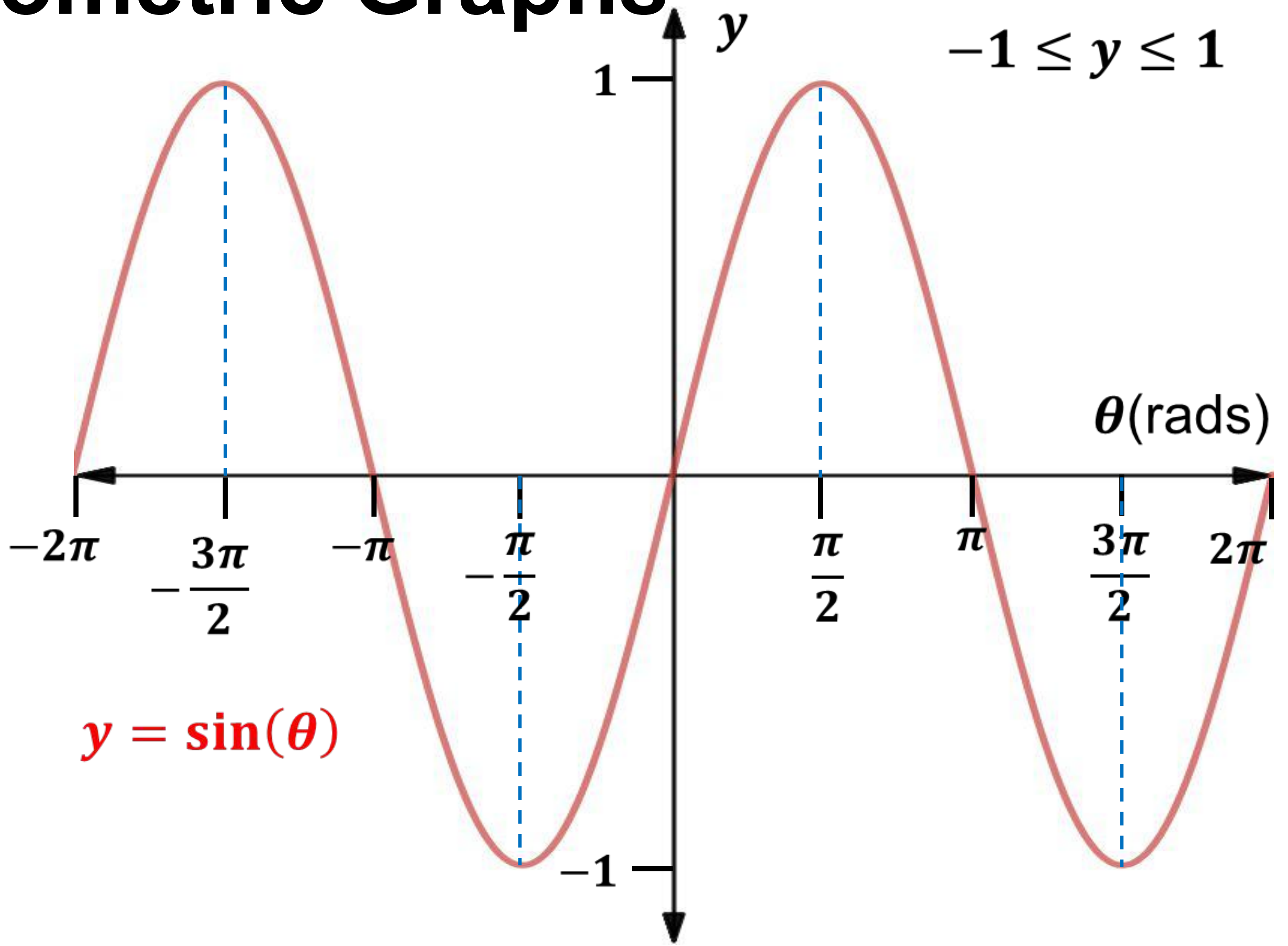


Trigonometric Graphs

The **graphs** of **trigonometric functions** are **periodic**.

- When working with **trigonometric functions** it can be helpful to **sketch** their **graphs** to easily see their **values** and **features**.

$$\sin(\theta) = \sin(\pi - \theta)$$



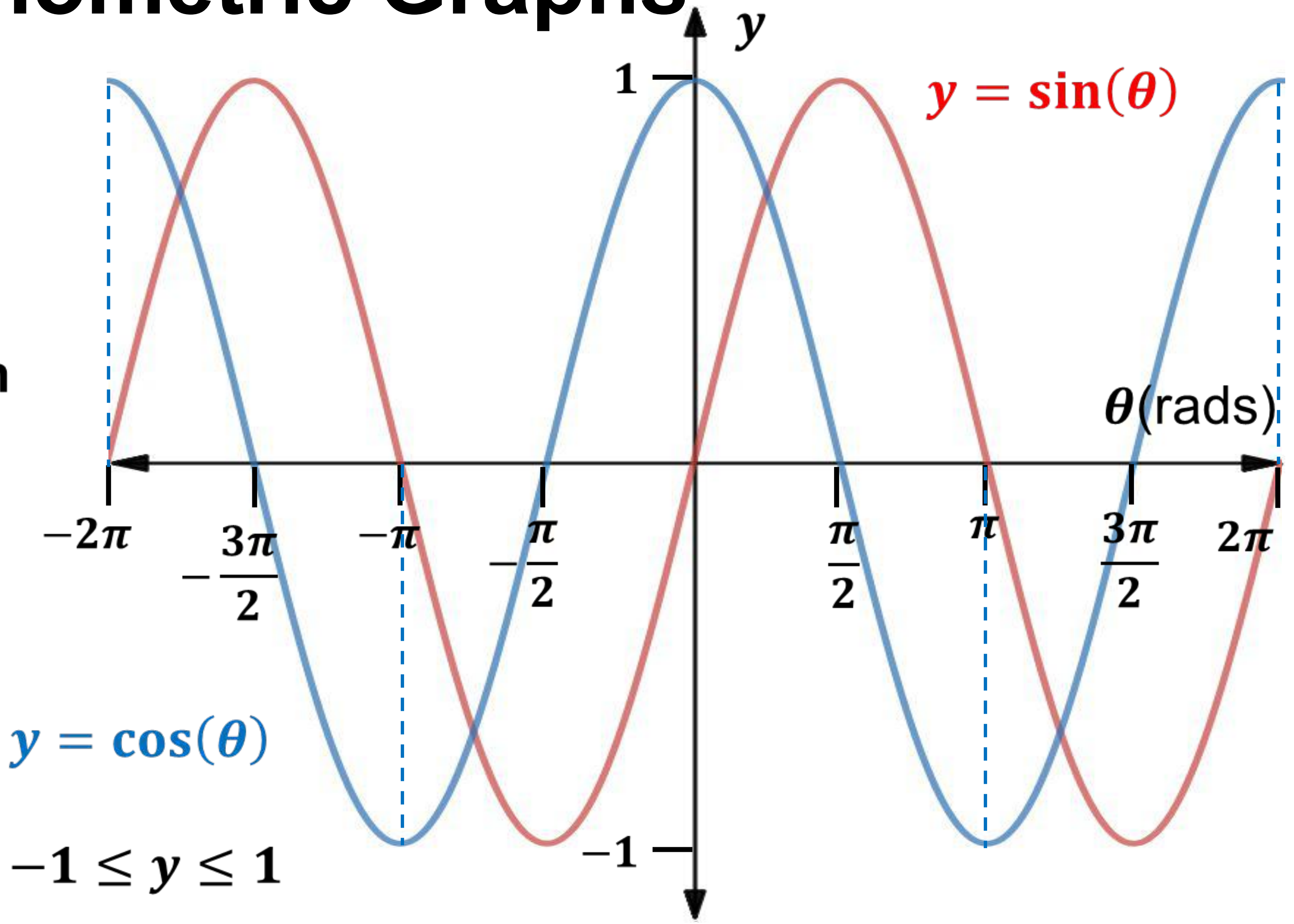
Trigonometric Graphs

$$\cos(\theta) = \cos(2\pi - \theta)$$

The **graph** of **cosine** is the **graph** of **sine translated** by $\frac{\pi}{2}$ **left**.

- This **property** can be used to **derive** the **identity**:

$$\sin(\theta) = \cos\left(\frac{\pi}{2} - \theta\right)$$

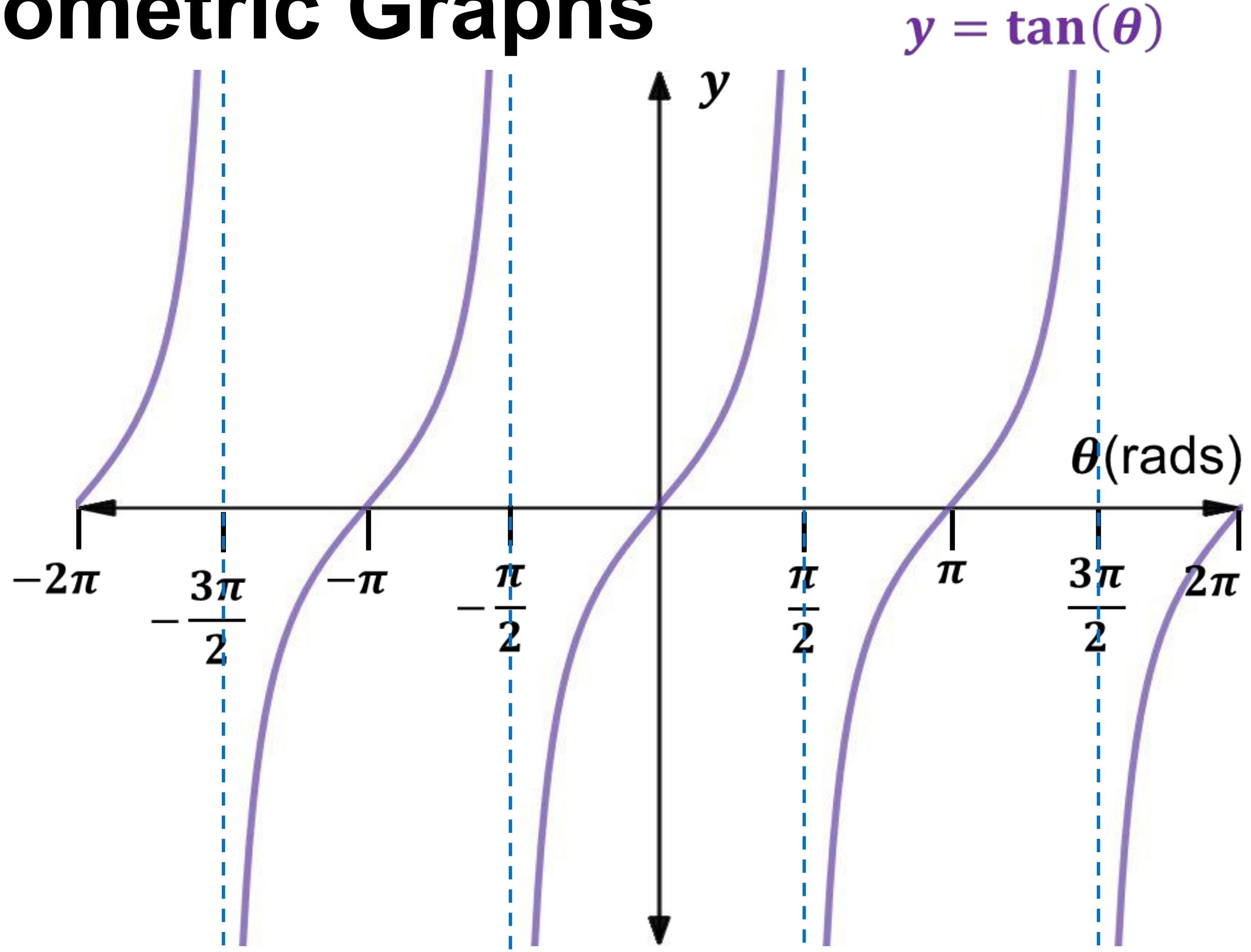


Trigonometric Graphs

The **graph** of **tangent** has an **infinite range**.

$$-\infty \leq y \leq \infty$$

- The graph has **asymptotes** at $\theta = \frac{\pi}{2} \pm n\pi$ where $y \to \infty$.



Trigonometric Graphs

Using these **graphs** we can find the **exact values** of the **trigonometric functions**.

	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0
	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0	1
	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	*N/A*	0	*N/A*	0

Compare to
$y = \sin(x)$

Exemplar Exam Question

1) The following figure shows a graph of the curve $y = a\sin(x + b)$ where $0 < b < \frac{\pi}{2}$.

(i) Given that the curve has a maximum at $\left(\frac{5\pi}{12}, \sqrt{3}\right)$ determine the values of a and b.

[2 marks]

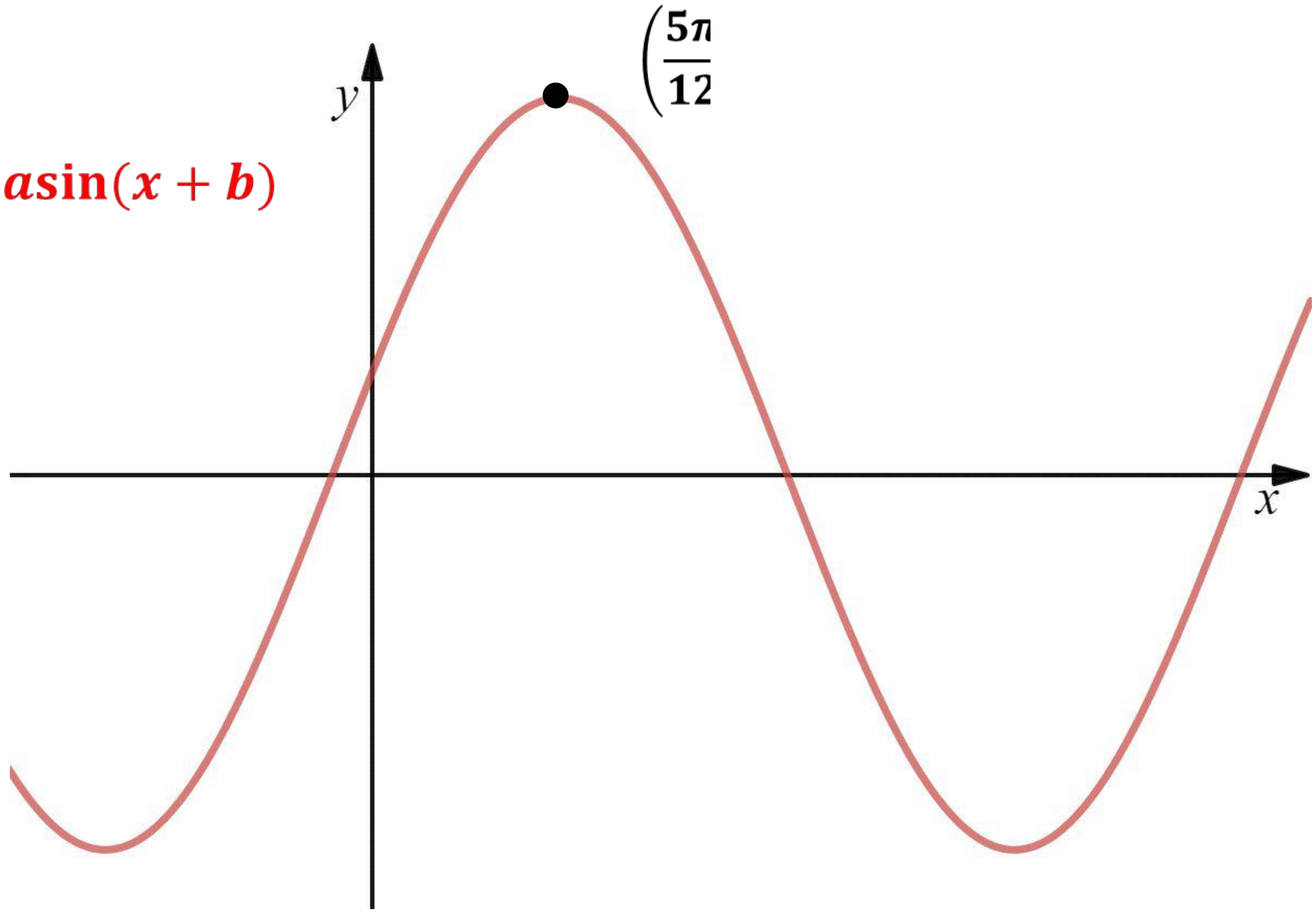
Key data.

(ii) Sketch the curve of $y = a\cos(x + b)$ on the same axes.

[1 mark]

Recall **properties** of **cosine graph**.

$y = a\sin(x + b)$



Exemplar Exam Question Answer

$y = a\sin(x + b)$ $\left(\frac{5\pi}{12}, \sqrt{3}\right)$

(i) Recall properties of sine function.

$f(x) = \sin(x)$ has maximum at $\left(\frac{\pi}{2}, 1\right)$

Compare y-values of maxima

$f(x)$ has maximum of 1, $af(x)$ has maximum of $\sqrt{3}$

$\Rightarrow a = \sqrt{3}$

[1 mark]

Exemplar Exam Question Answer

$y = \sqrt{3}\sin(x+b)$ $\left(\frac{5\pi}{12}, \sqrt{3}\right)$

Compare x-values of maxima

$f(x)$ has maximum at $\frac{\pi}{2}$, $f(x+b)$ has maximum at $\frac{5\pi}{12}$

$$\frac{\pi}{2} = \frac{5\pi}{12} + b$$

$$\Rightarrow b = \frac{\pi}{12}$$

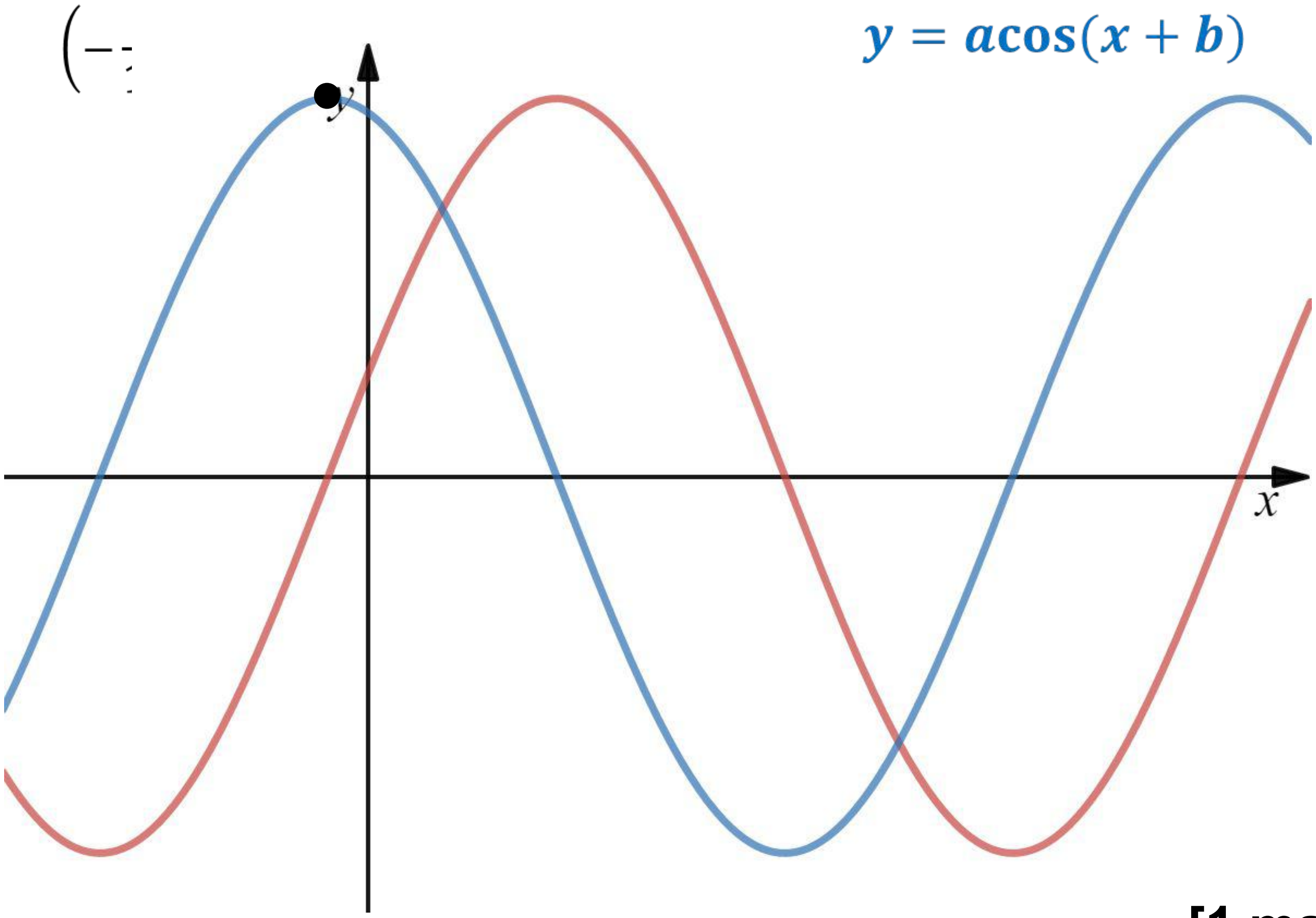
[1 mark]

Exemplar Exam Question Answer

(ii) Recall relation between sine and cosine.

The graph of cosine is the graph of sine translated by $\frac{\pi}{2}$ left.

$y = a\cos(x + b)$



[1 mark]

Advanced Trigonometric Functions

$\arcsin\theta$ $\sec\theta$

Specification Points - AQA

	Content
E4	Understand and use the definitions of secant, cosecant and cotangent and of arcsin, arccos and arctan; their relationships to sine, cosine and tangent; understanding of their graphs; their ranges and domains.

	Content
E5	Understand and use $\tan\theta \equiv \frac{\sin\theta}{\cos\theta}$ Understand and use $\sin^2\theta + \cos^2\theta \equiv 1$; $\sec^2\theta \equiv 1 + \tan^2\theta$ and $\text{cosec}^2\theta \equiv 1 + \cot^2\theta$

	Content
E7	Solve simple trigonometric equations in a given interval, including quadratic equations in sin, cos and tan and equations involving multiples of the unknown angle.

	Content
E8	Construct proofs involving trigonometric functions and identities.

Specification Points – OCR A

h) Understand and be able to use the definitions of secant ($\sec\theta$), cosecant ($\operatorname{cosec}\theta$) and cotangent ($\cot\theta$) and of $\arcsin\theta$, $\arccos\theta$ and $\arctan\theta$ and their relationships to $\sin\theta$, $\cos\theta$ and $\tan\theta$ respectively.

i) Understand the graphs of the functions given in 1.05h, their ranges and domains.

In particular, learners should know that the principal values of the inverse trigonometric relations may be denoted by $\arcsin\theta$ or $\sin^{-1}\theta$, $\arccos\theta$ or $\cos^{-1}\theta$, $\arctan\theta$ or $\tan^{-1}\theta$ and relate their graphs (for the appropriate domain) to the graphs of $\sin\theta$, $\cos\theta$ and $\tan\theta$.

o) Be able to solve simple trigonometric equations in a given interval, including quadratic equations in $\sin\theta$, $\cos\theta$ and $\tan\theta$ and equations involving multiples of the unknown angle.

e.g.
$\sin\theta = 0.5$ for $0 \le \theta < 360°$
$6\sin^2\theta + \cos\theta - 4 = 0$ for $0 \le \theta < 360°$
$\tan 3\theta = -1$ for $-180° < \theta < 180°$

p) Be able to construct proofs involving trigonometric functions and identities.

e.g. Prove that

$\cos^2\left(\theta + 45°\right) - \frac{1}{2}(\cos 2\theta - \sin 2\theta) = \sin^2\theta.$

Includes constructing a mathematical argument as described in Section 1.01.

j) Understand and be able to use $\tan\theta \equiv \frac{\sin\theta}{\cos\theta}$ and $\sin^2\theta + \cos^2\theta \equiv 1$.

In particular, these identities may be used in solving trigonometric equations and simple trigonometric proofs.

k) Understand and be able to use $\sec^2\theta \equiv 1 + \tan^2\theta$ and $\operatorname{cosec}^2\theta \equiv 1 + \cot^2\theta$.

In particular, the identities in 1.05j and 1.05k may be used in solving trigonometric equations, proving trigonometric identities or in evaluating integrals.

Specification Points – OCR MEI

t13	Understand and use the definitions of the sec, cosec and cot functions.	Including knowledge of the angles for which they are undefined.
t14	Understand relationships between the graphs of the sin, cos, tan, cosec, sec and cot functions.	Including domains and ranges.
t15	Understand and use the relationships $\tan^2\theta + 1 = \sec^2\theta$ and $\cot^2\theta + 1 = \operatorname{cosec}^2\theta$.	

t19	Use trigonometric identities, relationships and definitions in solving equations.
t20	Construct proofs involving trigonometric functions and identities.

Specification Points - Edexcel

5.4	Understand and use the definitions of secant, cosecant and cotangent and of arcsin, arccos and arctan; their relationships to sine, cosine and tangent; understanding of their graphs; their ranges and domains.	Angles measured in both degrees and radians.
5.5	**Understand and use** $\tan\theta = \frac{\sin\theta}{\cos\theta}$ **Understand and use** $\sin^2\theta + \cos^2\theta = 1$ $\sec^2\theta = 1 + \tan^2\theta$ and $\text{cosec}^2\theta = 1 + \cot^2\theta$	**These identities may be used to solve trigonometric equations** and angles may be in degrees or radians. **They may also be used to prove further identities.**
5.7	**Solve simple trigonometric equations in a given interval, including quadratic equations in sin, cos and tan and equations involving multiples of the unknown angle.**	**Students should be able to solve equations such as** $\sin(x + 70°) = 0.5$ for $0 < x < 360°$, $3 + 5\cos 2x = 1$ for $-180° < x < 180°$ $6\cos^2 x + \sin x - 5 = 0$, $0 \leqslant x < 360°$ These may be in degrees or radians and this will be specified in the question.
5.8	Construct proofs involving trigonometric functions and identities.	Students need to prove identities such as $\cos x \cos 2x + \sin x \sin 2x \equiv \cos x$.

Trigonometric Equations

Inverse trigonometric functions can be used to **solve trigonometric equations**.

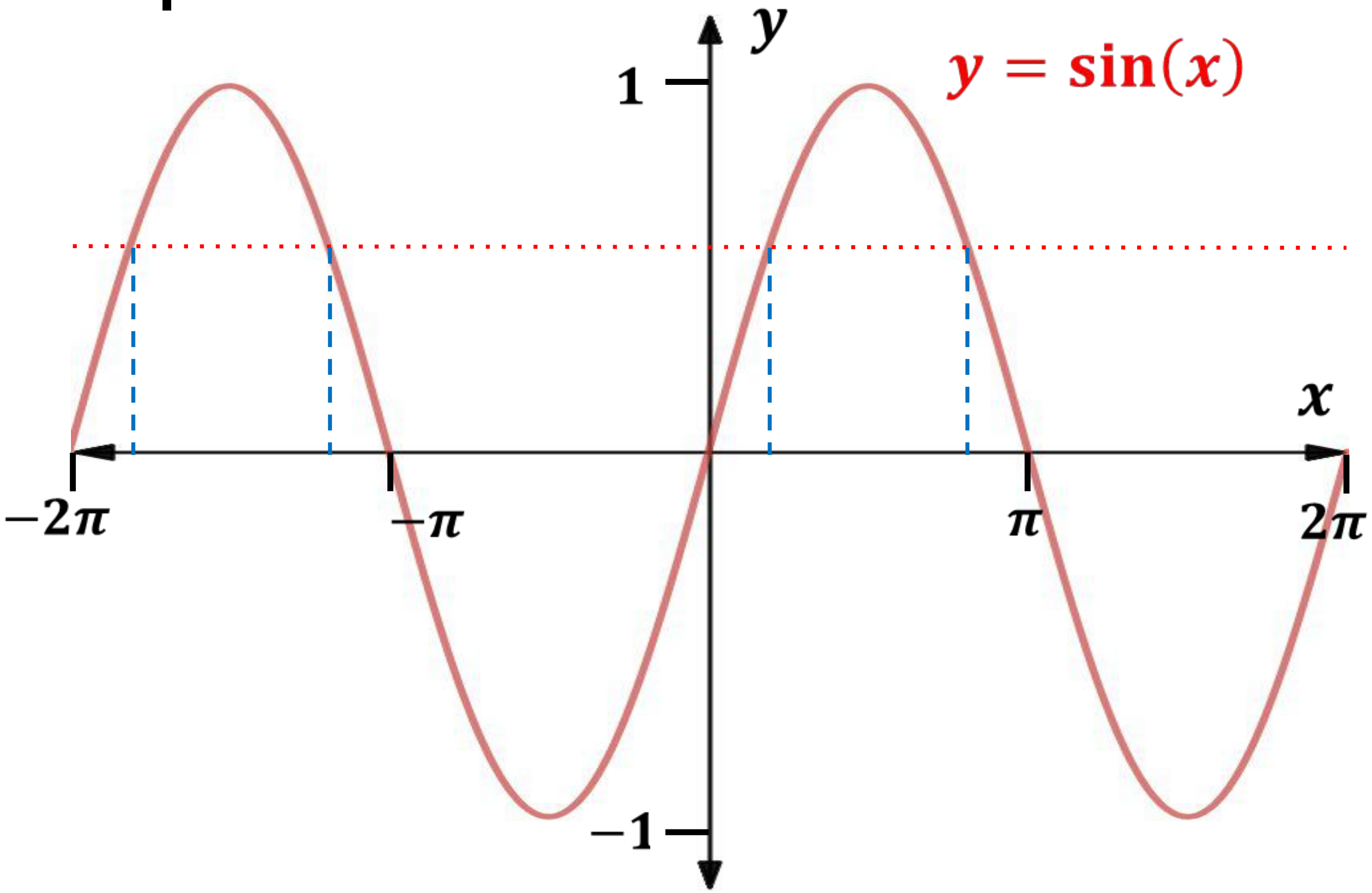
$$\sin^{-1} x = \arcsin(x) \qquad \cos^{-1} x = \arccos(x) \qquad \tan^{-1} x = \arctan(x)$$

- Using a **calculator**, these **functions** will **return** a **single value** known as the **principal value**.

- When asked to give an **answer** in a certain **range** there may be **multiple values** as **sin**, **cos** and **tan** are **periodic**.

Trigonometric Equations

To find the **other values** consider the **trigonometric graphs** and their **symmetries** and **periodicities**.



$\sin(x) = a \quad \Rightarrow \quad x = \sin^{-1}(a)$

Find x in **range** $-2\pi < x < 2\pi$

Trigonometric Equations

Sometimes we will need to **solve equations** of the **form** $\sin(ax+b)=c$ for x where x is in the **range** $x_{\mathrm{L}}<x<x_{\mathrm{U}}$.

- Let $\theta=ax+b$ and solve for θ in the range $\theta_{\mathrm{L}}<\theta<\theta_{\mathrm{U}}$.

Exemplar Exam Question

1) Find the exact solutions to the following equation for $\boldsymbol{\theta}$ in the range $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

$$\cos\left(3x + \frac{\pi}{4}\right) = \frac{1}{2}$$

May be **more than one solution**.

[3 marks]

Solutions should be **exact values**, most likely **fractions** in terms of $\boldsymbol{\pi}$.

Exemplar Exam Question Answer

$$\cos\left(3x + \frac{\pi}{4}\right) = \frac{1}{2}$$

Use inverse cosine function to find principal value.

$$\left(3x + \frac{\pi}{4}\right) = \arccos\left(\frac{1}{2}\right)$$

$$\left(3x + \frac{\pi}{4}\right) = \frac{\pi}{3}$$ or …

[1 mark]

Exemplar Exam Question Answer

Find new range for all possible solutions.

$$-\frac{\pi}{2} < x < \frac{\pi}{2}$$

Let $\theta = 3x + \frac{\pi}{4}$

$$-\frac{3\pi}{2} + \frac{\pi}{4} < \theta < \frac{3\pi}{2} + \frac{\pi}{4}$$

$$-\frac{5\pi}{4} < \theta < \frac{7\pi}{4}$$

Exemplar Exam Question Answer

$$-\frac{5\pi}{4} < \theta < \frac{7\pi}{4}$$

Use properties of cosine graph to find solutions in range.

$\cos(\theta) = \cos(2\pi - \theta)$ Period $= 2\pi$

$\theta = \frac{\pi}{3}$ or … $\left(2\pi - \frac{\pi}{3}\right) = \frac{5\pi}{3}$ or … $\left(\frac{5\pi}{3} - 2\pi\right) = -\frac{\pi}{3}$

[1 mark]

Exemplar Exam Question Answer

Determine x.

$$\left(3x+\frac{\pi}{4}\right)=\frac{\pi}{3} \quad \text{or} \ldots \quad \frac{5\pi}{3} \quad \text{or} \ldots \quad -\frac{\pi}{3}$$

$$3x=\frac{\pi}{12} \quad \text{or} \ldots \quad \frac{17\pi}{12} \quad \text{or} \ldots \quad -\frac{7\pi}{12}$$

$$x=\frac{\pi}{36} \quad \text{or} \ldots \quad \frac{17\pi}{36} \quad \text{or} \ldots \quad -\frac{7\pi}{36}$$

[1 mark]

Reciprocal Trigonometric Functions

Reciprocal trigonometric functions are **equal** to **1 divided** by a **trigonometric function**.

$$\mathbf{cosec}(\theta) = \frac{1}{\sin(\theta)}$$

$$\sec(\theta) = \frac{1}{\cos(\theta)}$$

$$\cot(\theta) = \frac{1}{\tan(\theta)}$$

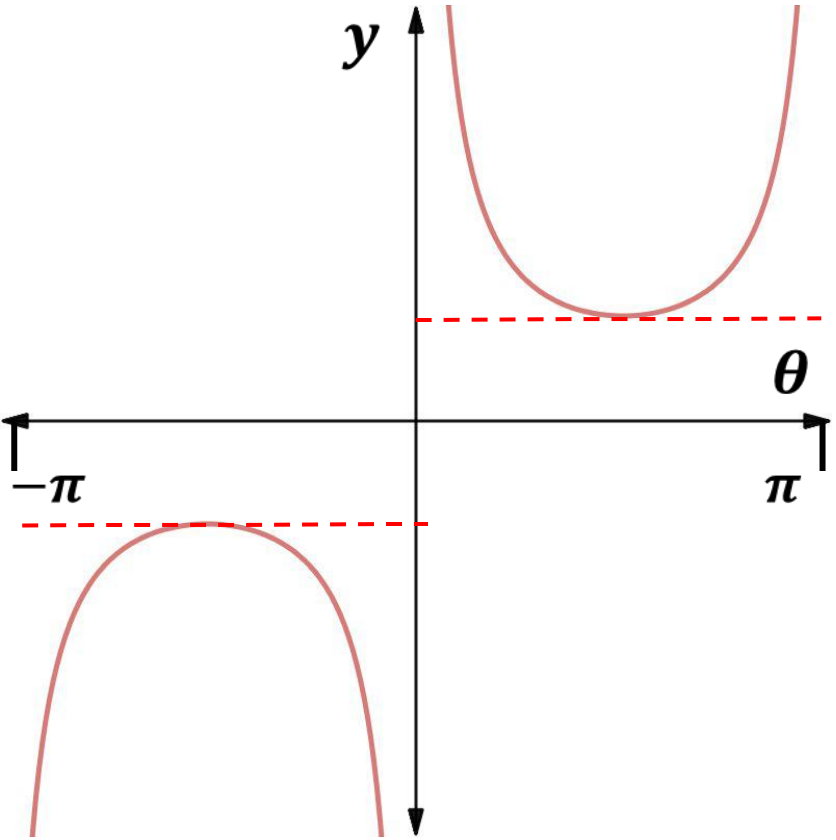
- These **functions** have the same **period** as their **respective trigonometric function**.

$$\sec^2(\theta) = 1 + \tan^2(\theta)$$

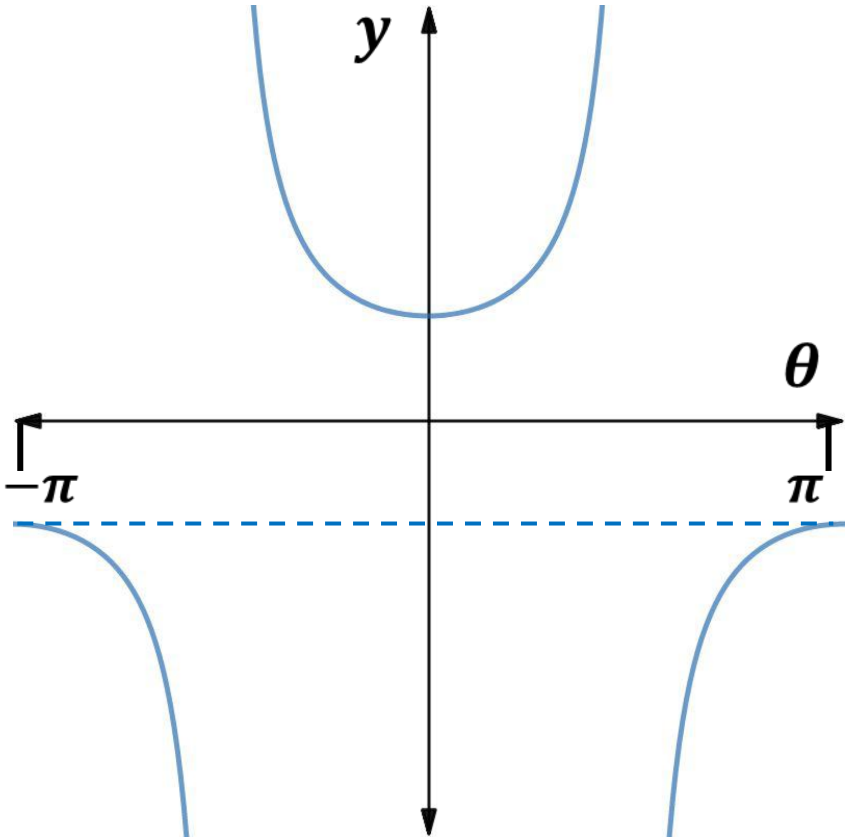
$$\mathrm{cosec}^2(\theta) = 1 + \cot^2(\theta)$$

Reciprocal Trigonometric Functions

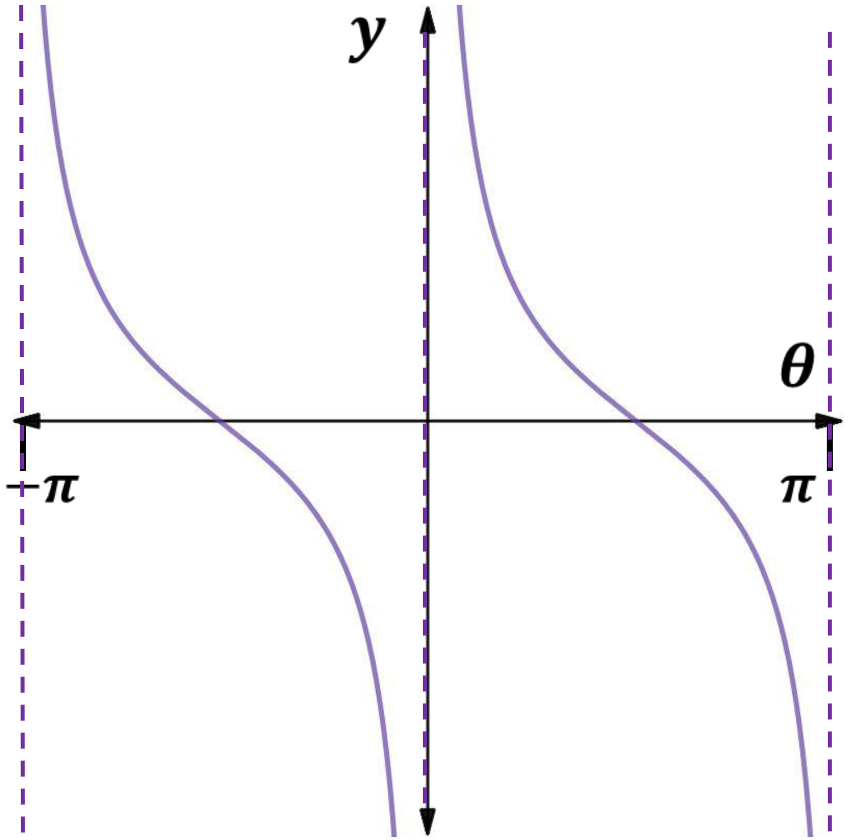
$y = \mathrm{cosec}(\theta)$



$y = \sec(\theta)$



$y = \cot(\theta)$



Reciprocal Trigonometric Functions

We will sometimes have to use **trigonometric identities** to solve **equations**.

Solve $2\tan^2(\theta) + 3\sec(\theta) = 0$ where $-\pi < \theta < \pi$

- **Identify** an **identity** to **simplify** the problem.

Exemplar Exam Question

1) By forming a quadratic in $\mathbf{cosec}\,x$ solve the following equation. Give your answers in terms of $\mathbf{sin}\,x$.

$$3\,\mathrm{cosec}\,x = \cot^2 x - 17$$

[3 marks]

Recall identities involving $\mathbf{cosec}\,x$ to replace $\mathbf{cot}^2\,x$ term.

Expect **solutions** of the **form** $\mathbf{sin}\,x$.

Exemplar Exam Question Answer

$3\operatorname{cosec} x = \cot^2 x - 17$

Use the identity: $\operatorname{cosec}^2(\theta) = 1 + \cot^2(\theta)$

$$\Rightarrow \cot^2(x) = \operatorname{cosec}^2(x) - 1$$

$3\operatorname{cosec} x = \operatorname{cosec}^2(x) - 1 - 17$

Form quadratic in $\operatorname{cosec} x$.

$\operatorname{cosec}^2(x) - 3\operatorname{cosec} x - 18 = 0$ **[1 mark]**

Exemplar Exam Question Answer

$$\text{cosec}^2(x) - 3\,\text{cosec}\,x - 18 = 0$$

Factorise and solve to find values of $\text{cosec}\,x$.

$$(\text{cosec}\,x - 6)(\text{cosec}\,x + 3) = 0$$

$\text{cosec}\,x = 6$ or -3 **[1 mark]**

Recall definition of $\text{cosec}\,x$.

$\text{cosec}\,x = \dfrac{1}{\sin x} \quad \Rightarrow \sin x = \dfrac{1}{6}$ or $-\dfrac{1}{3}$ **[1 mark]**

Compound Angles

$$\sin^2(\alpha + \beta)$$

Specification Points - AQA

	Content
E6	Understand and use double angle formulae; use of formulae for $\sin(A \pm B)$, $\cos(A \pm B)$ and $\tan(A \pm B)$; understand geometrical proofs of these formulae. Understand and use expressions for $a\cos\theta + b\sin\theta$ in the equivalent forms of $r\cos(\theta \pm \alpha)$ or $r\sin(\theta \pm \alpha)$

Specification Points – OCR A

l) Understand and be able to use double angle formulae and the formulae for $\sin(A \pm B)$, $\cos(A \pm B)$ and $\tan(A \pm B)$.

Learners may be required to use the formulae to prove trigonometric identities, simplify expressions, evaluate expressions exactly, solve trigonometric equations or find derivatives and integrals.

m) Understand the geometrical proofs of these formulae.

n) Understand and be able to use expressions for $a\cos\theta + b\sin\theta$ in the equivalent forms of $R\cos(\theta \pm \alpha)$ or $R\sin(\theta \pm \alpha)$.

In particular, learners should be able to:

1. sketch graphs of $a\cos\theta + b\sin\theta$,

2. determine features of the graphs including minimum or maximum points and

3. solve equations of the form $a\cos\theta + b\sin\theta = c$.

Specification Points – OCR MEI

Mt16	Understand and use the identities for $\sin(\theta \pm \phi)$, $\cos(\theta \pm \phi)$, $\tan(\theta \pm \phi)$.	Includes understanding geometric proofs. The starting point for the proof will be given.
t17	Know and use identities for $\sin 2\theta$, $\cos 2\theta$, $\tan 2\theta$.	Includes understanding derivations from $\sin(\theta + \phi)$, $\cos(\theta + \phi)$, $\tan(\theta + \phi)$. $\cos 2\theta = \cos^2\theta - \sin^2\theta$ $\cos 2\theta = 2\cos^2\theta - 1$ $\cos 2\theta = 1 - 2\sin^2\theta$
t18	Understand and use expressions for $a\cos\theta \pm b\sin\theta$ in the equivalent forms $R\sin(\theta \pm \alpha)$ and $R\cos(\theta \pm \alpha)$.	Includes sketching the graph of the function, finding its maximum and minimum values and solving equations.

Specification Points - Edexcel

5.6	Understand and use double angle formulae; use of formulae for $\sin(A \pm B)$, $\cos(A \pm B)$, and $\tan(A \pm B)$, understand geometrical proofs of these formulae.	To include application to half angles. Knowledge of the $\tan(\frac{1}{2}\theta)$ formulae will *not* be required.
	Understand and use expressions for $a\cos\theta + b\sin\theta$ in the equivalent forms of $r\cos(\theta \pm \alpha)$ or $r\sin(\theta \pm \alpha)$	Students should be able to solve equations such as $a\cos\theta + b\sin\theta = c$ in a given interval.

Compound Angle Formulae

Compound Angle Formulae expand **trigonometric functions** to **expressions** involving **sums** and **differences**.

$$\sin(\alpha \pm \beta) = \sin(\alpha) \times \cos(\beta) \pm \cos(\alpha) \times \sin(\beta)$$

$$\cos(\alpha \pm \beta) = \cos(\alpha) \times \cos(\beta) \mp \sin(\alpha) \times \sin(\beta)$$

$$\tan(\alpha \pm \beta) = \frac{\tan(\alpha) \pm \tan(\beta)}{1 \mp \tan(\alpha) \times \tan(\beta)}$$

Double Angle Formulae

To determine the **double angle formulae** substitute $\beta = \alpha$ into the **compound angle formulae**:

- $\sin(2\alpha) = 2 \times \sin(\alpha)\cos(\alpha)$

- $\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha)$

  $\cos(2\alpha) = 2\cos^2(\alpha) - 1$

  $\cos(2\alpha) = 1 - 2\sin^2(\alpha)$

- $\tan(2\alpha) = \dfrac{2 \times \tan(\alpha)}{1 - \tan^2(\alpha)}$

Exemplar Exam Question

1) Prove the following identity:

$$\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta} \equiv \sec 2\theta + \tan 2\theta$$

[3 marks]

Lay out **each step clearly** as **marks** are **awarded** for **working**.

Recall double angle formulae.

Exemplar Exam Question Answer

$$\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta} = \sec 2\theta + \tan 2\theta$$

Begin with more complicated side of identity.

$$\text{R.H.S} = \sec 2\theta + \tan 2\theta$$

$$= \frac{1}{\cos 2\theta} + \frac{\sin 2\theta}{\cos 2\theta}$$

$$= \frac{1 + \sin 2\theta}{\cos 2\theta}$$

Exemplar Exam Question Answer

Recall double angle formulae:

- $\sin(2\alpha) = 2 \times \sin(\alpha)\cos(\alpha)$
- $\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha)$

$$\Rightarrow \text{R.H.S} = \frac{1 + 2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta}$$

[1 mark]

Exemplar Exam Question Answer

$$\text{L.H.S} = \frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta} \qquad \text{R.H.S} = \frac{1 + 2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta}$$

Want to manipulate numerator of R.H.S to create terms which will cancel with denominator of R.H.S to give L.H.S.

Substitute: $1 = \cos^2\theta + \sin^2\theta$

$$\text{R.H.S} = \frac{\cos^2\theta + \sin^2\theta + 2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta}$$

[1 mark]

Exemplar Exam Question Answer

Factorise top and bottom to look for factors which will cancel.

$$\text{R.H.S} = \frac{\cos^2\theta + 2\sin\theta\cos\theta + \sin^2\theta}{\cos^2\theta - \sin^2\theta}$$

$$= \frac{(\cos\theta + \sin\theta)(\cos\theta + \sin\theta)}{(\cos\theta + \sin\theta)(\cos\theta - \sin\theta)}$$

$$= \frac{(\cos\theta + \sin\theta)}{(\cos\theta - \sin\theta)}$$

$$= \text{L.H.S}$$

[1 mark]

Exemplar Exam Question Answer

ALTERNATIVELY, begin with L.H.S and multiply top and bottom of fraction by $\cos\theta + \sin\theta$.

$$\text{L.H.S} = \frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}$$

$$= \frac{(\cos\theta + \sin\theta)(\cos\theta + \sin\theta)}{(\cos\theta - \sin\theta)(\cos\theta + \sin\theta)}$$

[1 mark]

Exemplar Exam Question Answer

$$\text{L.H.S} = \frac{(\cos\theta + \sin\theta)(\cos\theta + \sin\theta)}{(\cos\theta - \sin\theta)(\cos\theta + \sin\theta)}$$

$$= \frac{\cos^2\theta + 2\cos\theta\sin\theta + \sin^2\theta}{\cos^2\theta - \sin^2\theta}$$

$$= \frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta - \sin^2\theta} + \frac{2\cos\theta\sin\theta}{\cos^2\theta - \sin^2\theta}$$

[1 mark]

Use trig identities for double angle and $\cos^2\theta + \sin^2\theta$

$$= \frac{1}{\cos 2\theta} + \frac{\sin 2\theta}{\cos 2\theta} = \sec 2\theta + \tan 2\theta$$

[1 mark]

Rearranging Trigonometric Expressions

The **compound angle formulae** can be used to **rearrange trigonometric expressions** to simpler **forms**.

$$f(x) = A\sin x + B\cos x \longrightarrow f(x) = R\sin(x + \alpha)$$

This can help us **solve** $f(x)$ or **sketch** it as some **transformation** of $y = \sin(x)$.

1. First use the **compound angle formula** to write:

$$R\sin(x + \alpha) = R\sin(x)\cos(\alpha) + R\cos(x)\sin(\alpha) \quad = a\sin(x) + b\cos(x)$$

Rearranging Trigonometric Expressions

2. **Form** an **equation** for A and B in terms of R and α:

$$A = R\cos\alpha \qquad B = R\sin\alpha$$

3. To find α, **determine** from these two **expressions** $\frac{b}{a}$:

Rearranging Trigonometric Expressions

$A = R\cos\alpha \qquad B = R\sin\alpha$

4. To find R, **determine** from these two **expressions** $A^2 + B^2$:

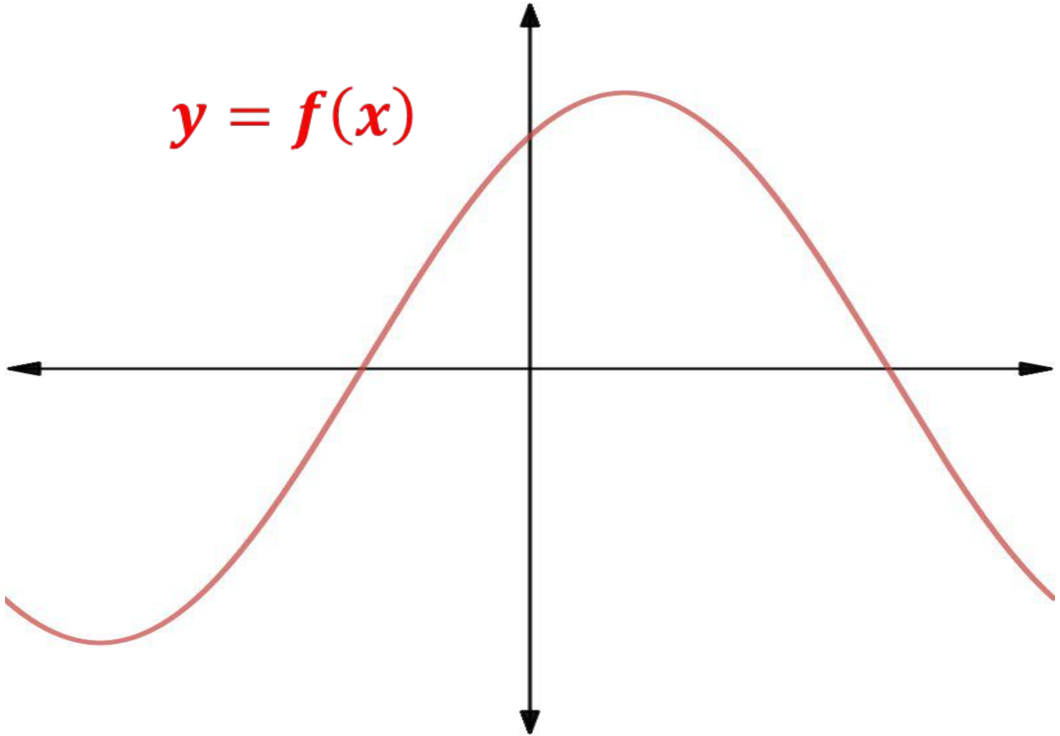
Rearranging Trigonometric Expressions

Therefore we can **rearrange** $f(x)$ to the **form**:

$$f(x) = A\sin x + B\cos x \longrightarrow f(x) = \sqrt{A^2 + B^2}\sin\left(x + \arctan\left(\frac{B}{A}\right)\right)$$

This helps us to:

- **Sketch** the graph of $f(x)$.
- **Solve** equations of the form $f(x) = c$.
- **Find** maximum and minimum values of $f(x)$.



Exemplar Exam Question

1) Solve the following equation for x in the range $-\pi \leq x \leq \pi$.

$$\cos x + \sqrt{3}\sin x = 2$$

May be **multiple solutions** for x.

[5 marks]

Can be **transformed** to **form** $R\sin(x+\alpha)$.

Exemplar Exam Question Answer

$\cos x + \sqrt{3}\sin x = 2$

Recall compound angle formula for $R\sin(x+\alpha)$.

$R\sin(x+\alpha) = R\sin(\alpha) \times \cos(x) + R\cos(\alpha) \times \sin(x)$

$= \cos x + \sqrt{3}\sin x$

$R\sin(\alpha) = 1$ $\quad\quad$ $R\cos(\alpha) = \sqrt{3}$ **[1 mark]**

Exemplar Exam Question Answer

Determine R:

$$R^2\cos^2(\alpha) + R^2\sin^2(\alpha) = \left(\sqrt{3}\right)^2 + 1^2$$

$$R^2(\cos^2(\alpha) + \sin^2(\alpha)) = 3 + 1$$

$$R^2 = 4$$

$$R = 2$$

[1 mark]

Exemplar Exam Question Answer

Determine α:

$$\frac{R\sin(\alpha)}{R\cos(\alpha)} = \frac{1}{\sqrt{3}}$$

$$\tan(\alpha) = \frac{1}{\sqrt{3}}$$

$$\alpha = \frac{\pi}{6}$$

[1 mark]

Exemplar Exam Question Answer

$$\cos x + \sqrt{3}\sin x = 2$$

Write equation in form: $R\sin(x + \alpha) = 2$

$$\cos x + \sqrt{3}\sin x = 2\sin\left(x + \frac{\pi}{6}\right)$$

$$2\sin\left(x + \frac{\pi}{6}\right) = 2$$

$$\sin\left(x + \frac{\pi}{6}\right) = 1$$

[1 mark]

Exemplar Exam Question Answer

Determine x in range $-\pi \leq x \leq \pi$.

$$\left(x + \frac{\pi}{6}\right) = \arcsin 1$$

$$x + \frac{\pi}{6} = \frac{\pi}{2}$$

$$x = \frac{\pi}{3}$$

[1 mark]

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Exam Question

1. Write the following expression in terms of $\sin(x)$.

$$f(x) = \sin(3x)$$

[2 marks]

2. Write the following expression in terms of $\cos(x)$.

$$f(x) = \cos(3x)$$

[2 marks]

Exam Question Answer

1. Apply angle addition formulae to $\sin(3x)$

$$\sin(3x) = \sin(2x + x)$$

$$= \sin 2x \cos x + \sin x \cos 2x$$

[1 mark]

$$= 2\sin x \cos^2 x + \sin x \left(2\cos^2 x - 1\right)$$

$$= 4\sin x \cos^2 x - \sin x$$

$$= 4\sin x \left(1 - \sin^2 x\right) - \sin x$$

$$= 3\sin x - 4\sin^3 x$$

[1 mark]

Exam Question Answer

2. Apply angle addition formulae to $\cos(3x)$

$$\cos(3x) = \cos(2x + x)$$

$$= \cos 2x \cos x - \sin 2x \sin x$$

[1 mark]

$$= \left(2\cos^2 x - 1\right)\cos x - 2\sin^2 x \cos x$$

$$= 2\cos^3 x - \cos x - 2\left(1 - \cos^2 x\right)\cos x$$

$$= 4\cos^3 x - 3\cos x$$

[1 mark]

Exam Question

3. (i) The function $f(x)$ is given by:

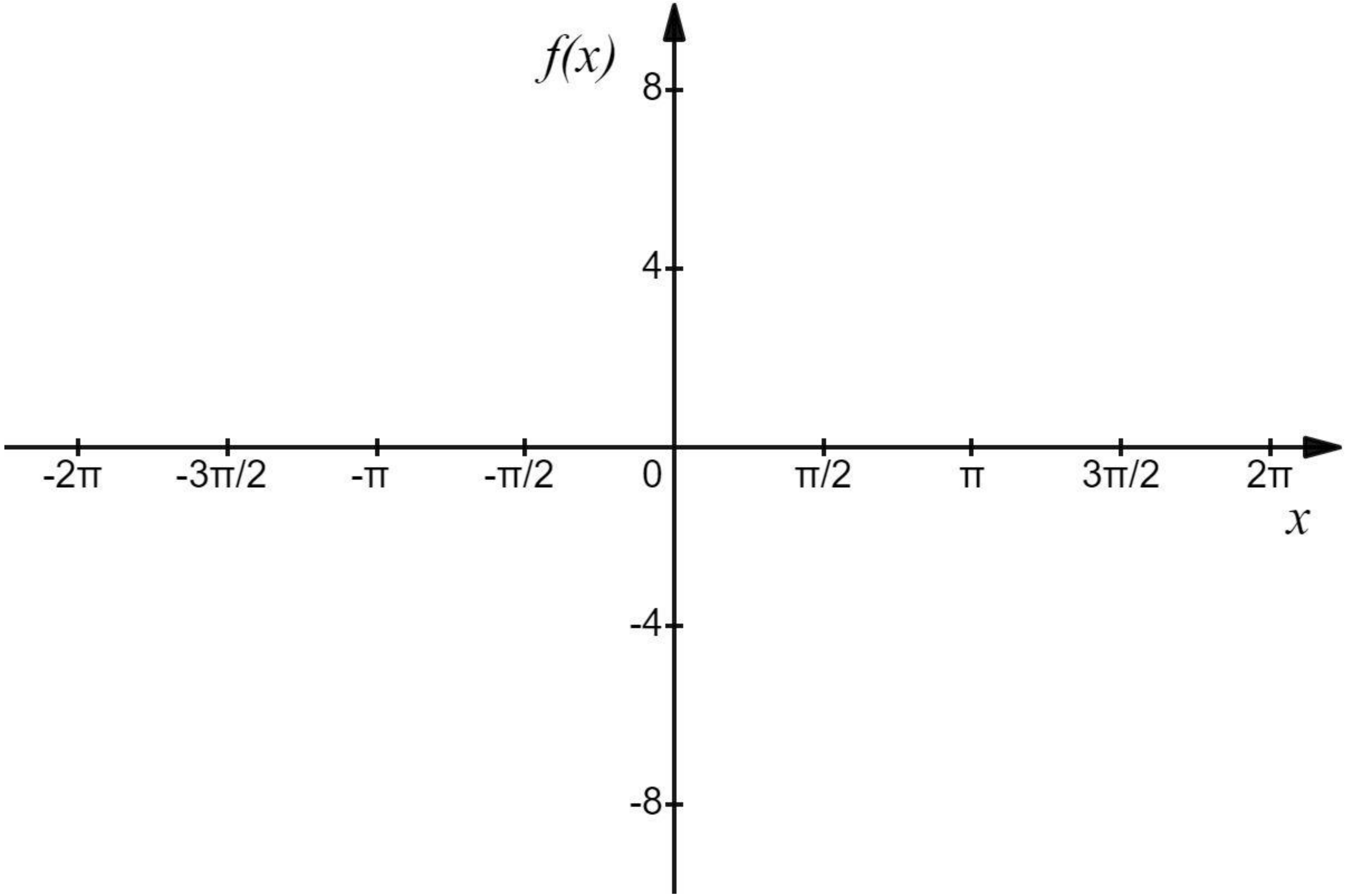
$$f(x) = 4\sin x - 6\cos x$$

Find the maximum value of $f(x)$, and the smallest value of x when this occurs. Give your answers to 3 s.f.

[6 marks]

(ii) Hence sketch $y = f(x)$ on the provided axes for the range $-2\pi \leq x \leq 2\pi$.

[2 marks]


f(x)
8
4
-4
-8
-2π
-3π/2
-π
-π/2
0
π/2
π
3π/2
2π
x

Exam Question Answer

3. (i) Compare $f(x)$ to e.g. $R\sin(x+\alpha)$

$$4\sin x - 6\cos x = R\sin x\cos\alpha + R\cos x\sin\alpha$$

[1 mark]

$$\Rightarrow R\sin\alpha = -6,\ R\cos\alpha = 4$$

[1 mark]

Calculate value of R

$$R^2(\sin^2\alpha + \cos^2\alpha) = (-6)^2 + 4^2$$

$$R = \sqrt{(-6)^2 + 4^2} = 2\sqrt{13} = 7.21$$

[1 mark]

Exam Question Answer

Calculate value of α

$$\tan\alpha = \frac{R\sin\alpha}{R\cos\alpha} = -\frac{6}{4}$$

$$\alpha = \tan^{-1}\left(-\frac{6}{4}\right) = -0.983$$

[1 mark]

Analyse converted form of $f(x)$

$$f(x) = 2\sqrt{13}\sin(x - 0.983) \Rightarrow f_{max} = 2\sqrt{13}$$

[1 mark]

This occurs when $x - 0.983 = \frac{\pi}{2} \Rightarrow x = 2.55$

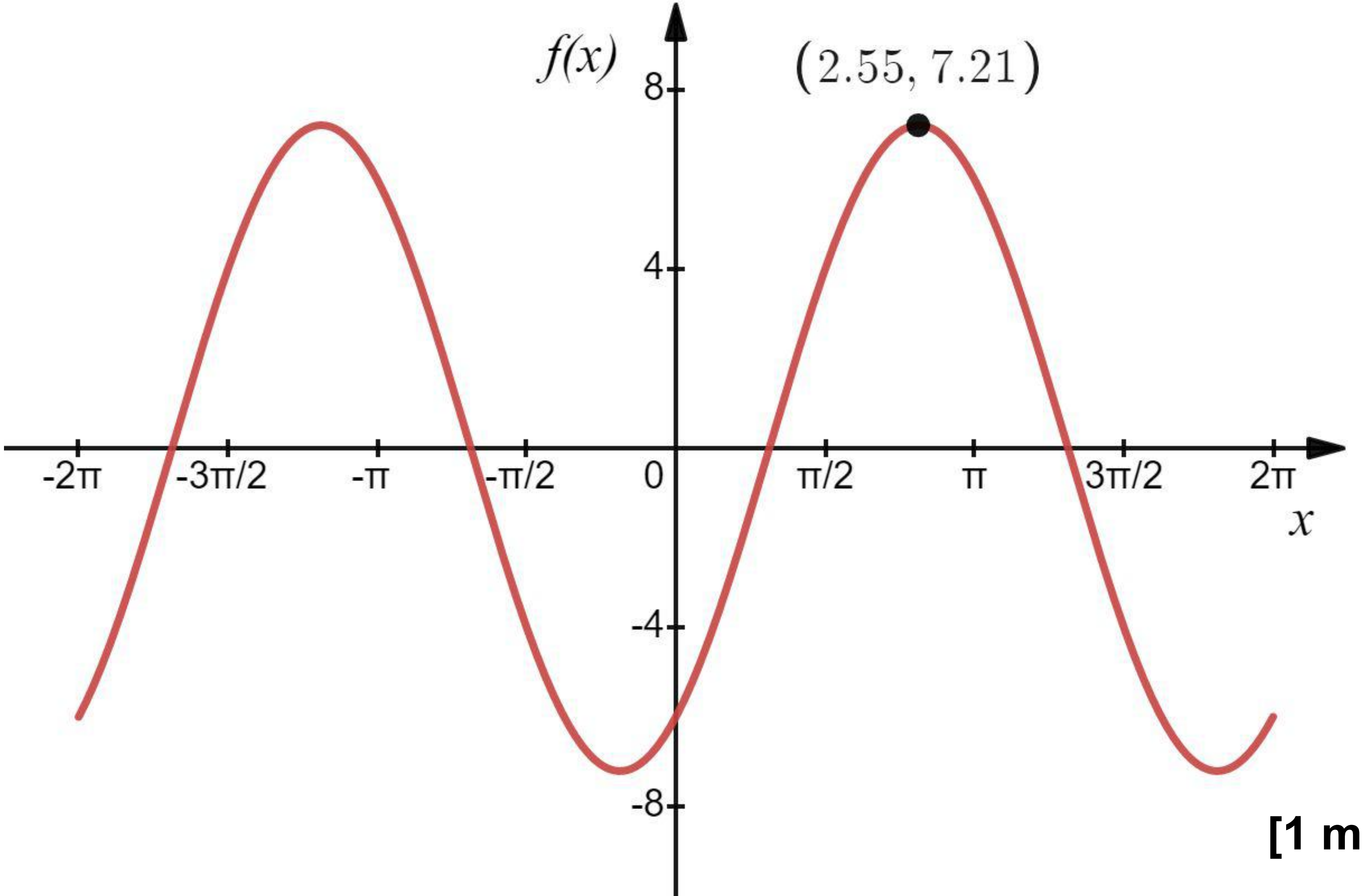
[1 mark]

Exam Question Answer

(ii) Deduce details of graph

$y = 7.21\sin(x - 0.983)$ is graph of $y = \sin x$ scaled vertically by 7.21 and translated to right by 0.983

[1 mark]



[1 mark]

Exam Question

4. Prove the following identity:

$$\frac{\sin x}{1 + \cos x} + \frac{1 + \cos x}{\sin x} = 2\,\mathrm{cosec}\,x$$

[3 marks]

Exam Question Answer

Begin with more complicated side of identity.

$$\text{L.H.S} = \frac{\sin x}{1+\cos x} + \frac{1+\cos x}{\sin x}$$

Express as single fraction over common denominator.

$$\text{L.H.S} = \frac{\sin x}{1+\cos x}\left(\times\frac{\sin x}{\sin x}\right) + \frac{1+\cos x}{\sin x}\left(\times\frac{1+\cos x}{1+\cos x}\right)$$

$$= \frac{\sin^2 x}{\sin x\,(1+\cos x)} + \frac{(1+\cos x)^2}{\sin x(1+\cos x)}$$

Exam Question Answer

$$\text{L.H.S} = \frac{\sin^2 x + (1 + \cos x)^2}{\sin x\,(1 + \cos x)}$$

[1 mark]

Multiply out bracket in numerator.

$$\text{L.H.S} = \frac{\sin^2 x + 1 + 2\cos x + \cos^2 x}{\sin x\,(1 + \cos x)} = \frac{\left(\sin^2 x + \cos^2 x\right) + 1 + 2\cos x}{\sin x\,(1 + \cos x)}$$

Use the identity: $\sin^2\theta + \cos^2\theta = 1$ $\Rightarrow \sin^2 x = 1 - \cos^2 x$

$$\text{L.H.S} = \frac{1 - \cos^2 x + 2\cos x + \cos^2 x}{\sin x\,(1 + \cos x)}$$

[1 mark]

Exam Question Answer

Simplify.

$$\text{L. H. S} = \frac{2 + 2\cos x}{\sin x\,(1 + \cos x)}$$

$$= \frac{2(1 + \cos x)}{\sin x\,(1 + \cos x)}$$

$$= \frac{2}{\sin x}$$

$$= 2\,\text{cosec}\, x \qquad = \text{R. H. S}$$

[1 mark]

Exam Question Answer

ALTERNATIVELY, Begin with more complicated side of identity.

This time, multiply top and bottom of left fraction by $1 - \cos x$.

$$\text{L.H.S} = \frac{\sin x}{1+\cos x}\left(\times \frac{1-\cos x}{1-\cos x}\right) + \frac{1+\cos x}{\sin x}$$

$$= \frac{\sin x\,(1-\cos x)}{(1+\cos x)(1-\cos x)} + \frac{1+\cos x}{\sin x}$$

[1 mark]

Exam Question Answer

Simplify.

$$\text{L.H.S} = \frac{\sin x\,(1-\cos x)}{(1+\cos x)(1-\cos x)} + \frac{1+\cos x}{\sin x}$$

$$= \frac{\sin x\,(1-\cos x)}{1-\cos^2 x} + \frac{1+\cos x}{\sin x}$$

[1 mark]

$$= \frac{1-\cos x}{\sin x} + \frac{1+\cos x}{\sin x}$$

$$= \frac{2}{\sin x} = 2\operatorname{cosec} x$$

[1 mark]