Pure Mathematics: Integration

$$\int f(x)\,dx = F(x) + c$$

$$\Rightarrow f(x) = \frac{d}{dx}F(x)$$

Material Covered

Applying Integration

1. Area Under a Curve
2. First Order Differential Equations

Integration Methods

1. Partial Fractions
2. Integration by Substitution
3. Integration by Parts

Simple Integrals

Specification Points - AQA

	Content
H1	Know and use the Fundamental Theorem of Calculus.

	Content
H2	Integrate x^n (excluding $n = -1$), and related sums, differences and constant multiples. Integrate e^{kx} , $\frac{1}{x}$, sin kx , cos kx and related sums, differences and constant multiples.

Specification Points – OCR A

Fundamental theorem of calculus	a) Know and be able to use the fundamental theorem of calculus. *i.e. Learners should know that integration may be defined as the reverse of differentiation and be able to apply the result that* $\int f(x)\,dx = F(x) + c \Leftrightarrow f(x) = \frac{d}{dx}(F(x))$, *for sufficiently well-behaved functions.* *Includes understanding and being able to use the terms indefinite and definite when applied to integrals.*	
Indefinite integrals	b) Be able to integrate x^n where $n \neq -1$ and related sums, differences and constant multiples. *Learners should also be able to solve problems involving the evaluation of a constant of integration e.g. to find the equation of the curve through* $(-1, 2)$ *for which* $\frac{dy}{dx} = 2x + 1$.	c) Be able to integrate e^{kx}, $\frac{1}{x}$, $\sin kx$, $\cos kx$ and related sums, differences and constant multiples. [*Integrals of* arcsin, arccos *and* arctan *will be given if required.*] *This includes using trigonometric relations such as double-angle formulae to facilitate the integration of functions such as* $\cos^2 x$.

Specification Points – OCR MEI

Mc19	Know that integration is the reverse of differentiation.	Fundamental Theorem of Calculus.
c20	Be able to integrate functions of the form kx^n where k is a constant and $n \neq -1$.	Including related sums and differences.
c21	Be able to find a constant of integration given relevant information.	e.g. Find y as a function of x given that $\frac{dy}{dx} = x^2 + 2$ and $y = 7$ when $x = 1$.

Mc24	Be able to integrate e^{kx}, $\frac{1}{x}$, $\sin kx$, $\cos kx$ and related sums, differences and constant multiples.	$\int \frac{1}{x}\,dx = \ln\lvert x \rvert + c, \quad x \neq 0$ x in radians for trigonometrical integrals.

Specification Points - Edexcel

8.1	**Know and use the Fundamental Theorem of Calculus**	**Integration as the reverse process of differentiation. Students should know that for indefinite integrals a constant of integration is required.**
8.2	**Integrate x^n (excluding $n = -1$) and related sums, differences and constant multiples.** Integrate e^{kx}, $\frac{1}{x}$, $\sin kx$, $\cos kx$ and related sums, differences and constant multiples.	**For example, the ability to integrate expressions such as $\frac{1}{2}x^2 - 3x^{-\frac{1}{2}}$ and $\frac{(x+2)^2}{x^{\frac{1}{2}}}$ is expected. x** **Given $f'(x)$ and a point on the curve, Students should be able to find an equation of the curve in the form $y = f(x)$.** To include integration of standard functions such as $\sin 3x$, $\sec^2 2x$, $\tan x$, e^{5x}, $\frac{1}{2x}$. Students are expected to be able to use trigonometric identities to integrate, for example, $\sin^2 x$, $\tan^2 x$, $\cos^2 3x$.

The Fundamental Theorem of Calculus

Integration is the **reverse process** of **differentiation**.

This is **shown** by the **Fundamental Theorem of Calculus:**

$$\int f(x)\,dx = F(x) + c \quad \Leftrightarrow \quad f(x) = \frac{d}{dx}F(x)$$

We can therefore **work out** the **integral** of a **function** if we know the **reverse derivative**.

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Exemplar Exam Question

How do we go from **gradient function** to **equation of a curve**?

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Why does it say a **full** equation?

What will this information be **useful** for?

Only **2 marks**, not much working involved.

Exemplar Exam Question Answer

Determine the integral of each term and put together

[1 Mark]

Exemplar Exam Question Answer

[1 Mark]

Integration of Trigonometric and Exponential Functions

We can use the fundamental theorem of calculus to deduce integrals of **trigonometric** and **exponential functions**.

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Integration of Trigonometric and Exponential Functions

The following **table** gives the **integrals** of **trigonometric functions** that will be useful to know:

Function			
Integral			
Function			
Integral			

Exemplar Exam Question

1) Evaluate the following integral:

Don't forget the **constant term**!

$$\int \frac{1}{x^2} + \frac{1}{x} - 2\sec^2 x \; dx$$

[2 marks]

All **relatively simple looking functions**, **testing basic knowledge** rather than **complex calculation skills**.

Only **2 marks** for **three terms**.

Exemplar Exam Question Answer

$$\int \frac{1}{x^2} + \frac{1}{x} - 2\sec^2 x \; dx$$

Integrate term by term using knowledge of standard differentials.

First term is polynomial, but constant in front is incorrect.

Need to include a constant in answer to compensate

Exemplar Exam Question Answer

$$-\frac{1}{x} + \int \frac{1}{x} - 2\sec^2 x \; dx$$

Integrate term by term using knowledge of standard differentials.

Next form looks like it follows the same rule, but is an exception

[1 Mark]

Exemplar Exam Question Answer

$$-\frac{1}{x} + \ln x + \int -2\sec^2 x \; dx$$

Integrate term by term using knowledge of standard differentials.

Last term is a trigonometric function you should recognise

Exemplar Exam Question Answer

$$\int \frac{1}{x^2} + \frac{1}{x} - 2\sec^2 x \; dx$$

$$= -\frac{1}{x} + \ln x - 2\tan x + c$$

[1 Mark]

Reverse Chain Rule

We can use the **reverse chain rule** to **integrate** more **complex functions**.

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Reverse Chain Rule

The rule can be extended to **other cases** of derivatives found with the chain rule

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Exemplar Exam Question

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[2 marks]

Can we use **exponential laws** to simplify this?

[2 marks]

How can we link this to a **standard integral**?

2 marks each

Exemplar Exam Question Answer

(i) Rearrange to obtain a function that is easier to integrate.

We can use properties of exponentials to simplify the integral.

$$\int \frac{(e^{x})^{2}}{e^{3x}}\,dx = \int \frac{e^{2x}}{e^{3x}}\,dx$$

[1 Mark]

Exemplar Exam Question Answer

Integrate using reverse chain rule

[1 Mark]

Exemplar Exam Question Answer

(ii) Rewrite into an easier to understand integral

$$\int x\sqrt{x^2+1}\,dx = \int x(x^2+1)^{\frac{1}{2}}\,dx$$

Exemplar Exam Question Answer

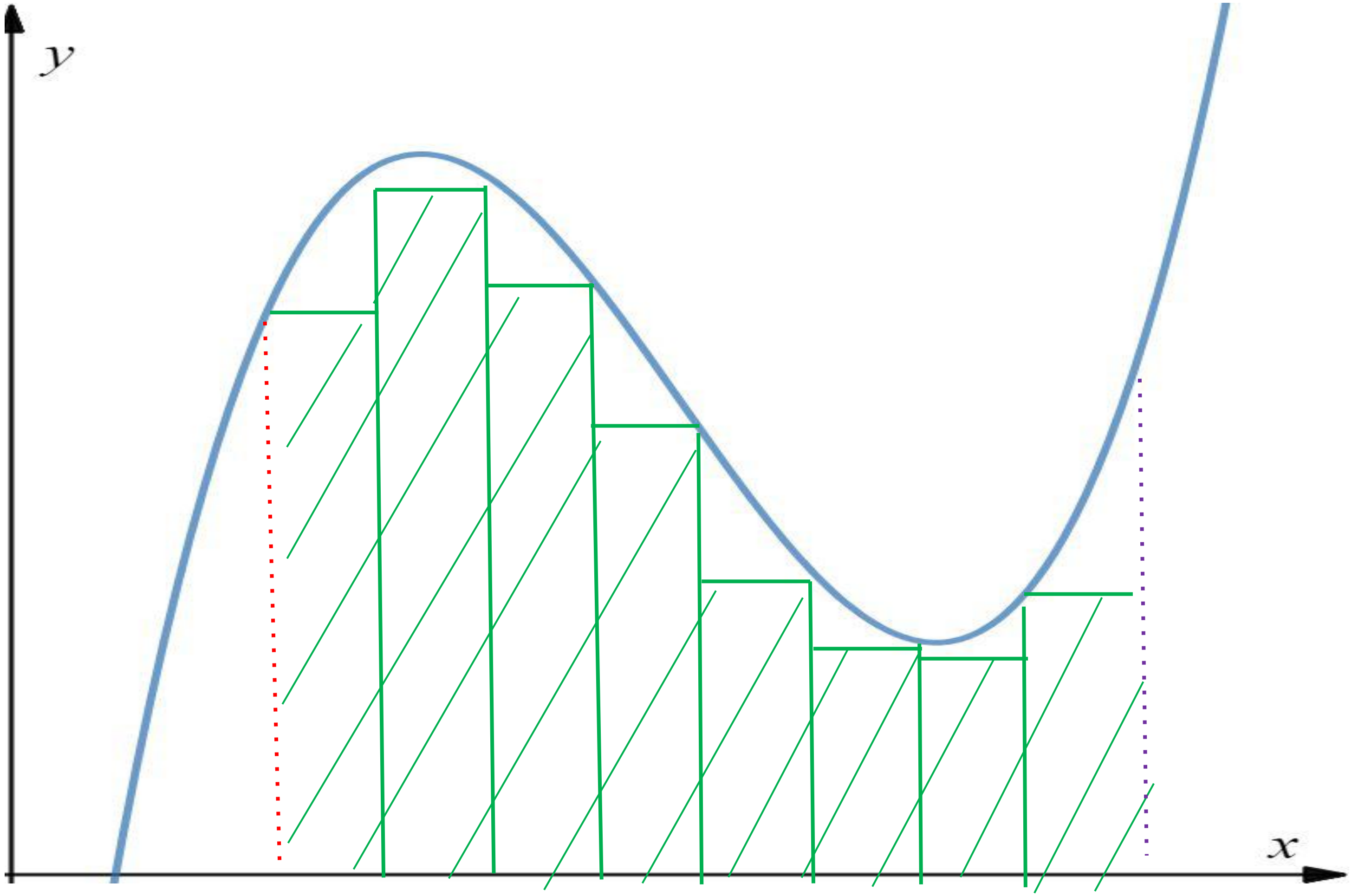
Apply the inverse chain rule

$$F(x) = \frac{2}{3}x^{\frac{3}{2}}$$

[1 Mark]

[1 Mark]

Applying Integration



Specification Points - AQA

	Content
H3	Evaluate definite integrals; use a definite integral to find the area under a curve and the area between two curves.

	Content
H4	Understand and use integration as the limit of a sum.

	Content
H7	Evaluate the analytical solution of simple first order differential equations with separable variables, including finding particular solutions (Separation of variables may require factorisation involving a common factor).

	Content
H8	Interpret the solution of a differential equation in the context of solving a problem, including identifying limitations of the solution; includes links to kinematics.

Specification Points – OCR A

		g) Understand and be able to use integration as the limit of a sum. *In particular, they should know that the area under a graph can be found as the limit of a sum of areas of rectangles.* *See also 1.09f.*	k) Be able to evaluate the analytical solution of simple first order differential equations with separable variables, including finding particular solutions. *Separation of variables may require factorisation involving a common factor.* *Includes: finding by integration the general solution of a differential equation involving separating variables or direct integration; using a given initial condition to find a particular solution.*
Definite integrals and areas	d) Be able to evaluate definite integrals. e) Be able to use a definite integral to find the area between a curve and the *x*-axis. *This area is defined to be that enclosed by a curve, the x-axis and two ordinates. Areas may be included which are partly below and partly above the x-axis, or entirely below the x-axis.*	f) Be able to use a definite integral to find the area between two curves. *This may include using integration to find the area of a region bounded by a curve and lines parallel to the coordinate axes, or between two curves or between a line and a curve.* *This includes curves defined parametrically.*	l) Be able to interpret the solution of a differential equation in the context of solving a problem, including identifying limitations of the solution. *Includes links to differential equations connected with kinematics.* *e.g. If the solution of a differential equation is $v = 20 - 20e^{-t}$, where v is the velocity of a parachutist, describe the motion of the parachutist.*

Specification Points – OCR MEI

c22	Know what is meant by indefinite and definite integrals. Be able to evaluate definite integrals.	e.g. $\int_1^3 (3x^2 + 5x - 1)\,dx$.
c23	Be able to use integration to find the area between a graph and the x-axis.	Includes areas of regions partly above and partly below the x-axis. General understanding that the area under a graph can be found as the limit of a sum of areas of rectangles.

Mc24	Be able to integrate e^{kx}, $\frac{1}{x}$, $\sin kx$, $\cos kx$ and related sums, differences and constant multiples.	$\int \frac{1}{x}\,dx = \ln\lvert x\rvert + c,\ \ x \neq 0$ x in radians for trigonometrical integrals.
c25	Understand integration as the limit of a sum.	Know that $\lim_{\delta x \to 0} \sum_a^b f(x)\,\delta x = \int_a^b f(x)\,dx$
c26	Be able to use integration to find the area between two curves.	Learners should also be able to find the area between a curve and the y-axis, including integrating with respect to y.
c31	Be able to formulate first order differential equations using information about rates of change.	Contexts may include kinematics, population growth and modelling the relationship between price and demand.
c32	Be able to find general or particular solutions of first order differential equations analytically by separating variables.	Equations may need to be factorised using a common factor before variables can be separated.
c33	Be able to interpret the solution of a differential equation in the context of solving a problem, including identifying limitations of the solution.	Includes links to kinematics.

Specification Points - Edexcel

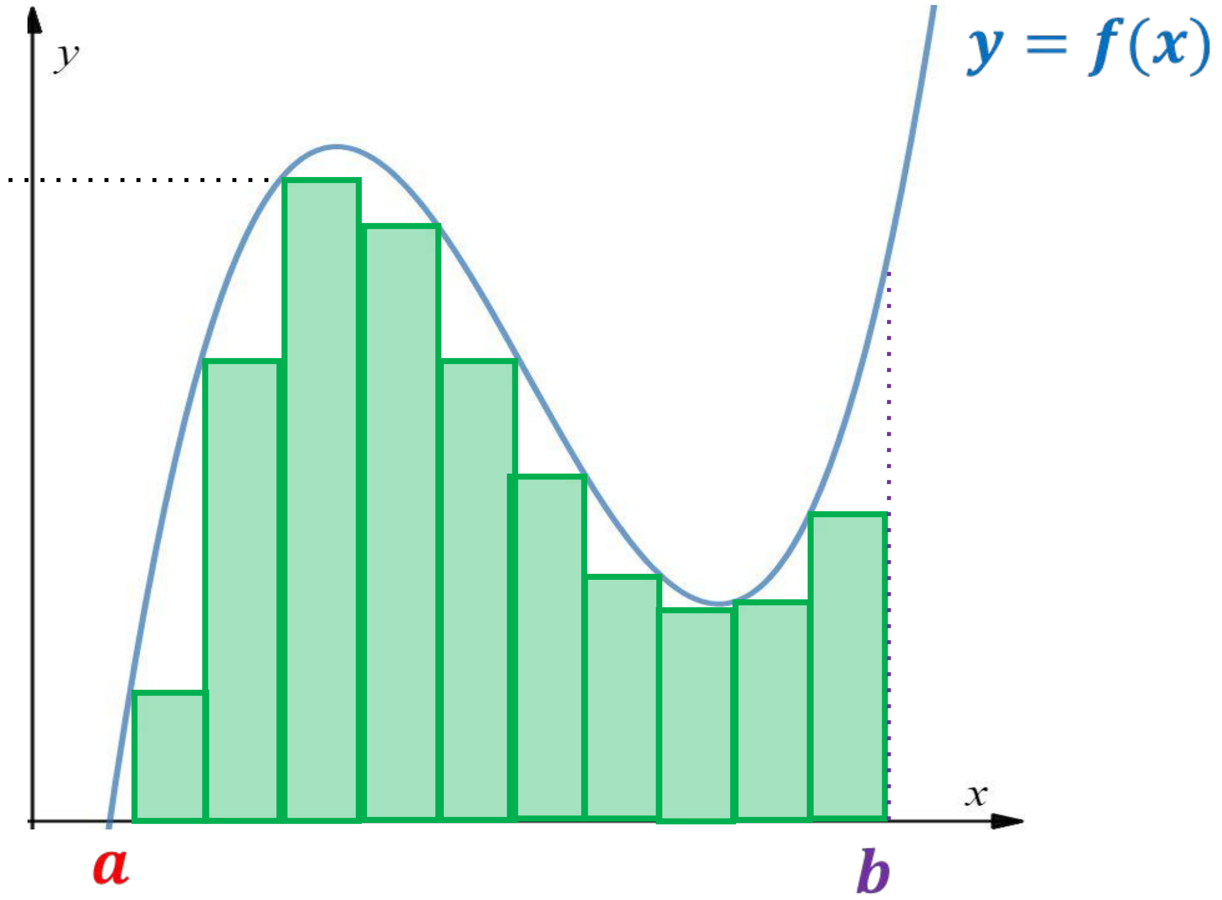
8.3	**Evaluate definite integrals; use a definite integral to find the area under a curve** and the area between two curves	Students will be expected to be able to evaluate the area of a region bounded by a curve and given straight lines, or between two curves. This includes curves defined parametrically. For example, find the finite area bounded by the curve $y = 6x - x^2$ and the line $y = 2x$ Or find the finite area bounded by the curve $y = x^2 - 5x + 6$ and the curve $y = 4 - x^2$.
8.4	Understand and use integration as the limit of a sum.	Recognise $\int_a^b f(x)\,dx = \lim_{\delta x \to 0} \sum_{x=a}^{b} f(x)\,\delta x$
8.7	Evaluate the analytical solution of simple first order differential equations with separable variables, including finding particular solutions (Separation of variables may require factorisation involving a common factor.)	Students may be asked to sketch members of the family of solution curves.
8.8	Interpret the solution of a differential equation in the context of solving a problem, including identifying limitations of the solution; includes links to kinematics.	The validity of the solution for large values should be considered.

Integration as the Limit of a Sum

Integrating a **function** is **equivalent** to **summing** the **area** of **rectangles** fitted **underneath** the **curve** of the **function**.

- As the **width** (δx) of these **rectangles** gets **narrower**, their **total area approaches** the **value** of the **integral**.

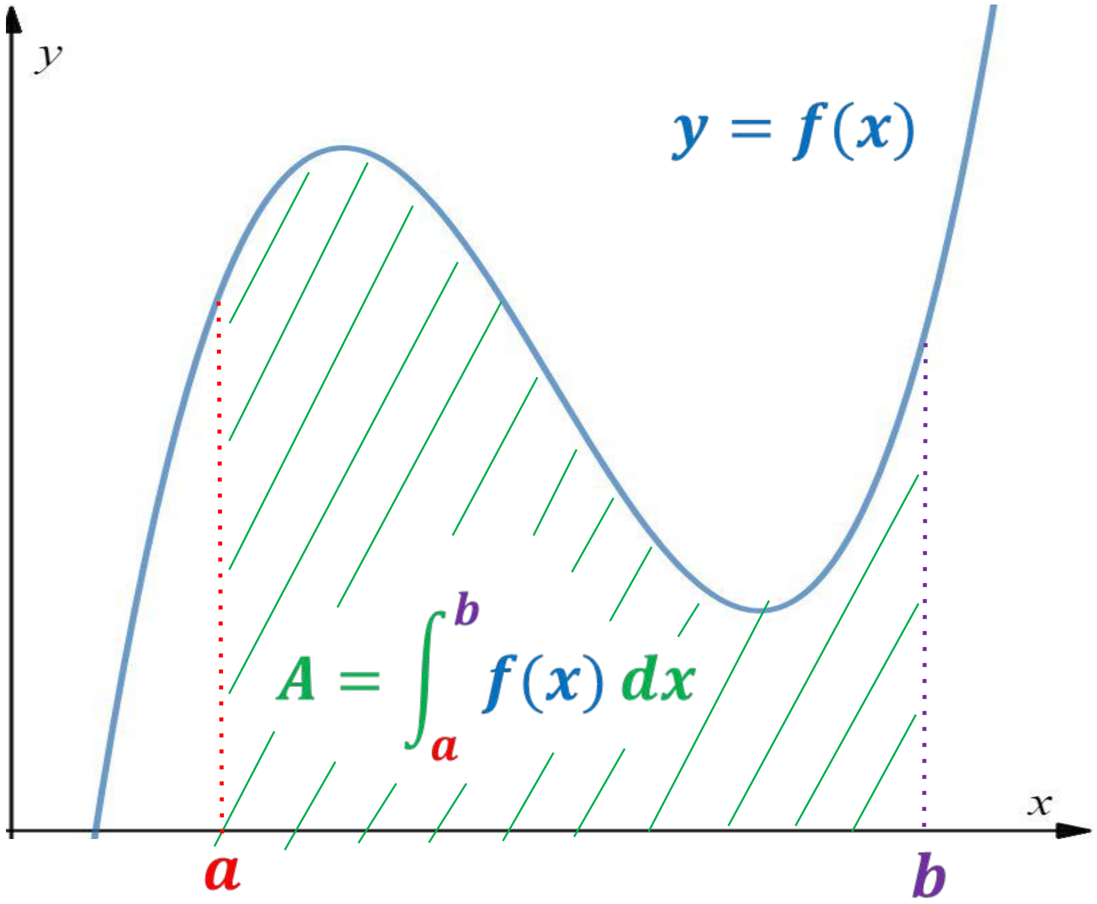
$$\int_a^b f(x)\,dx = \lim_{\delta x \to 0} \sum_{x=a}^{b} f(x)\,\delta x$$



Integration as the Area Under a Curve

Therefore, **integrating** a **function** gives the **area** between its **curve** and the x-axis between the **limits** a and b.

$$A = \int_a^b f(x)\,dx$$

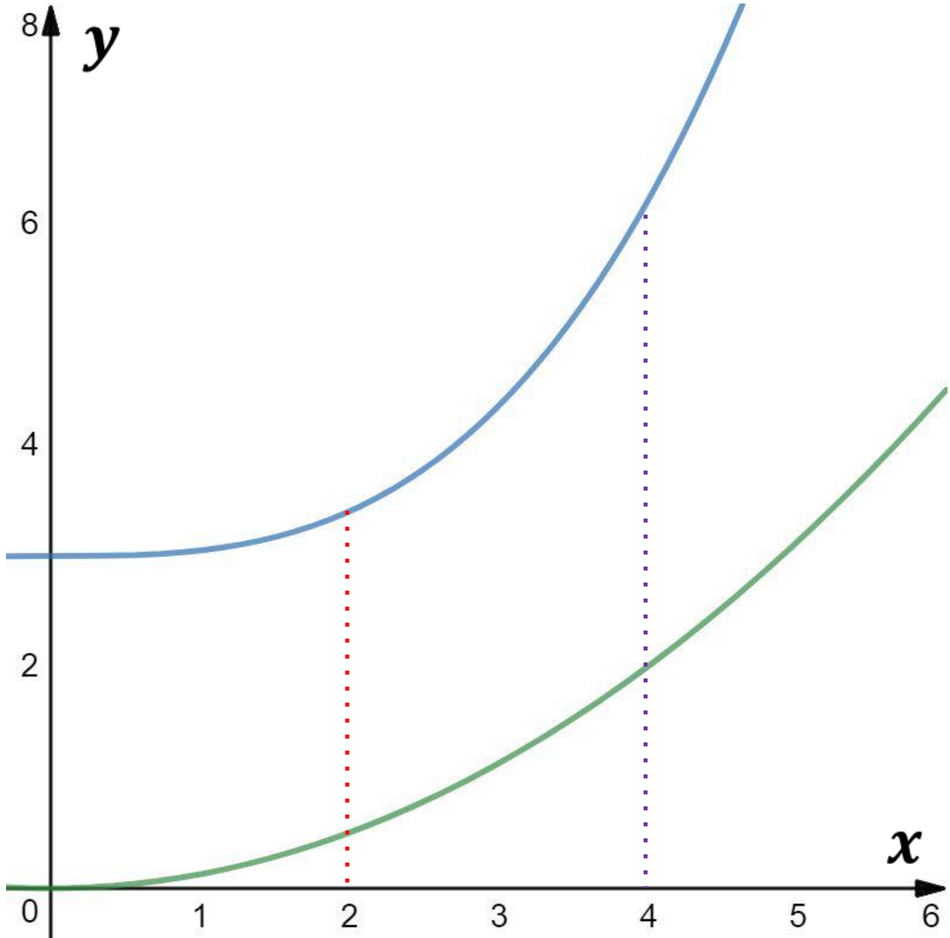


- A **definite integral** has **limits**.
- There is no need to include the **constant** c.

Area Between 2 Curves

We can **determine** the **area** between **2 curves** $y = f(x)$ and $y = g(x)$ between certain limits a and b.

1. Find the **area** under **each curve** between the **limits**.



2. Subtract the **area** under the **lower curve**.

How to find **areas under curves** and **difference** between them

Exemplar Exam Question

1) Calculate the area, A, between the curves

$$f(x) = x^2 + 2$$

$$g(x) = \frac{1}{(x+2)^2}$$

between $x = 0$ and $x = 2$. You should show all your working and leave your answer as a single fraction.

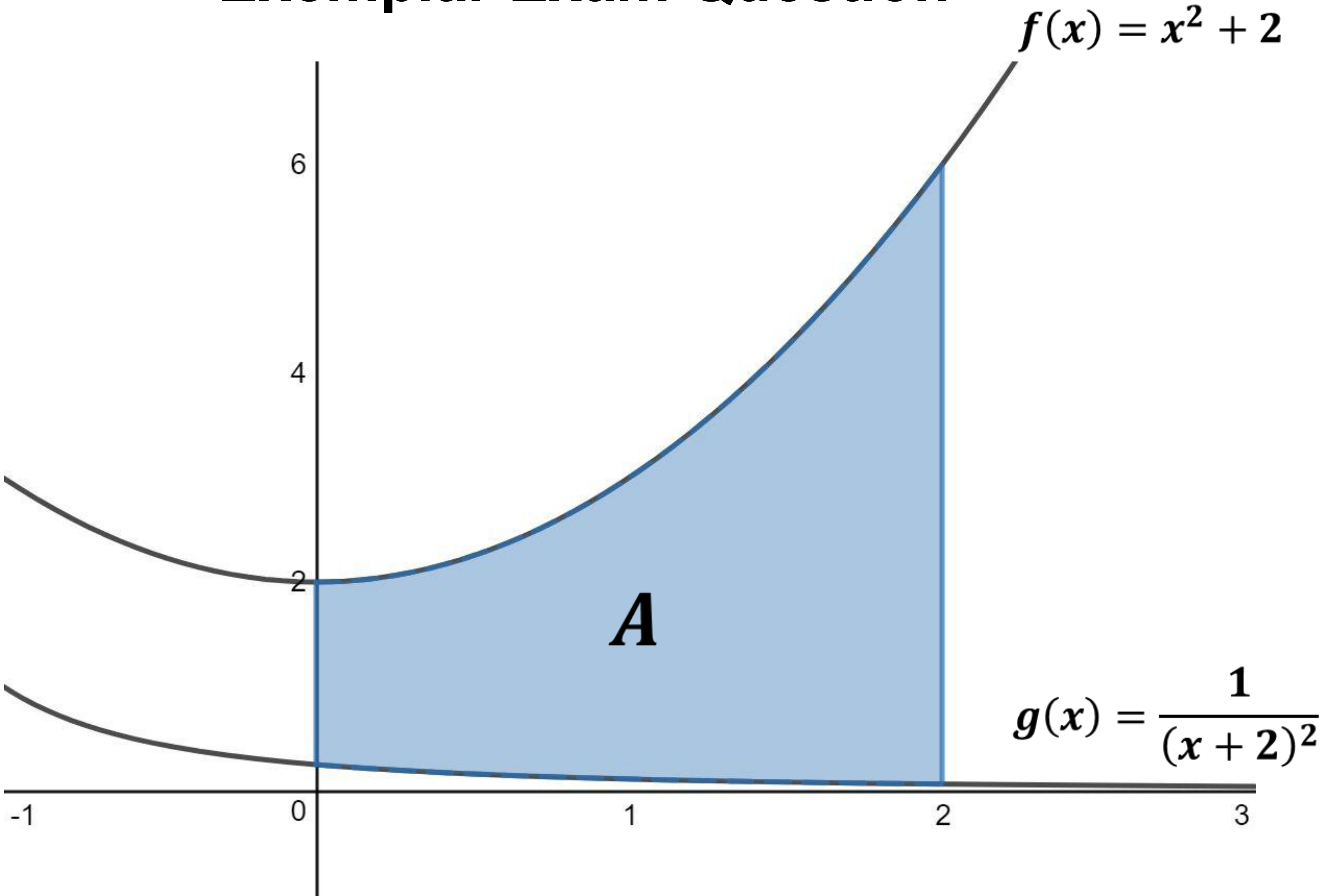
[5 marks]

Think back to **standard integrals**, what ones apply here?

Cannot just plug into calculator!

5 mark question, **5 key steps**. **Two for each integral** and **one** for finding **difference**

Exemplar Exam Question



Exemplar Exam Question Answer

$f(x) = x^2 + 2$ $g(x) = \frac{1}{(x+2)^2}$

Find area under first curve, A_1

Area under curve given by $f(x)$, A_1, given by it's integral with limits 0 and 2

$$A_1 = \int_0^2 x^2 + 2\,dx$$

Integral is solved using standard integral $\int x^n dx = \frac{1}{n+1} x^{n+1}$

$$A_1 = \left[\frac{1}{3}x^3 + 2x\right]_0^2$$

[1 Mark]

Exemplar Exam Question Answer

$$A_1 = \left[\frac{1}{3}x^3 + 2x\right]_0^2 \qquad g(x) = \frac{1}{(x+2)^2}$$

Now plug in limits for x

$$A_1 = \left[\frac{1}{3}(2)^3 + 2(2)\right] - \left[\frac{1}{3}(0)^3 + 2(0)\right]$$

$$= \left[\frac{8}{3} + 4\right] - [0 + 0]$$

$$= \frac{20}{3}$$

[1 Mark]

Exemplar Exam Question Answer

$$A_1 = \frac{20}{3} \qquad g(x) = \frac{1}{(x+2)^2}$$

Find area under second curve, A_2

Area under curve given by $g(x)$, A_2, given by it's integral with limits 0 and 2

$$A_2 = \int_0^2 \frac{1}{(x+2)^2}\,dx$$

Integral is solved using standard integral $\int \frac{1}{(x+a)^n}\,dx = -\frac{1}{(n-1)(x+a)^{n-1}}$

$$A_2 = \left[-\frac{1}{x+2}\right]_0^2$$

[1 Mark]

Exemplar Exam Question Answer

$$A_1 = \frac{20}{3}$$

$$A_2 = \left[-\frac{1}{x+2}\right]_0^2$$

Now plug in limits for x

$$A_2 = \left[-\frac{1}{2+2}\right] - \left[-\frac{1}{0+2}\right]$$
$$= \left[-\frac{1}{4}\right] - \left[-\frac{1}{2}\right]$$
$$= \frac{1}{4}$$

[1 Mark]

Exemplar Exam Question Answer

$A_1 = \frac{20}{3}$ $A_2 = \frac{1}{4}$

Find difference between areas

Area between curves, A, is the difference between areas under curves

$$A = A_1 - A_2$$

$$= \frac{20}{3} - \frac{1}{4}$$

$$= \frac{77}{12}$$

[1 Mark]

First Order Differential Equations

We can use **integration** to **solve equations** of the **form**:

$$\frac{dy}{dx} = f(x) \times g(y)$$

- Begin by **separating variables** (x and y) onto **different sides** of the **equation**.

$$\frac{1}{g(y)} \times \frac{dy}{dx} = f(x)$$

- **Multiply** by dx and **integrate** both **sides**.

$$\int \frac{1}{g(y)}\, dy = \int f(x)\, dx$$

First Order Differential Equations

We can **rearrange** the **result** to find the **general solution** of y in terms of x.

- To find the **particular solution**, we need to **determine** the **value** of the **integration constant** c.
- To do this we **require** a **particular value** (x, y).
- We can **write** $A = e^c$.

Questions are indicating **what kind of answers you should expect**, good way of checking

Exemplar Exam Question

Lot of information, what **differential equations** are we working with?

1) Certain stars rapidly expand at the end of their life. One star expands in such a way that its surface area, $\boldsymbol{S}$, increases at a rate given by $\boldsymbol{2ktS}$, where $\boldsymbol{k}$ is a proportionality constant and $\boldsymbol{t}$ is time in seconds. For the following question, you may treat the star as spherical

(i) Find the general solution to (i) in the form $\boldsymbol{S = Ae^{\alpha t^{\beta}}}$, clearly stating the values of the constants $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$.

[3 Marks]

(ii) Given that $\boldsymbol{k = 2.65\ s^{-2}}$ and the initial radius of the star is $\boldsymbol{1.39 \times 10^{14}\ m}$, use your solution to calculate the surface area of the star after $\boldsymbol{1}$ second to 3 significant figures.

[2 Marks]

What does **initial radius** mean in terms of our solution?

One **long 5-mark question** broken down to **2 parts**

Exemplar Exam Question Answer

(i) Derive rate of change of S

Use information in question to derive an equation for $\frac{dS}{dt}$

Told in question that surface area S increases at a rate given by $2ktS$.

$$\Rightarrow \frac{dS}{dt} = 2ktS$$

Exemplar Exam Question Answer

$$\frac{dS}{dt} = 2ktS$$

Separate S and t terms

Can divide by S to separate variables

$$\frac{1}{S}\frac{dS}{dt} = 2kt$$

[1 Mark]

Form integral on both sides.

Multiply by dt and integrate both sides.

$$\int \frac{1}{S}\, dS = \int 2kt\, dt$$

Exemplar Exam Question Answer

$$\int \frac{1}{S}\, dS = \int 2kt\, dt$$

Solve integrals

LHS is of form $\frac{1}{x}$, so integral is $\ln x$

RHS is of form ax^n, so integral is $\frac{a}{n+1}x^{n+1}$

Integrate both sides. Remember to include a constant term.

$\ln S = kt^2 + c$ $\quad$ c constant term

[1 Mark]

Exemplar Exam Question Answer

$$\ln S = kt^2 + c$$

Rearrange to correct form: $S = Ae^{\alpha t^{\beta}}$

Take exponentials of both sides

$$S = e^{kt^2 + c}$$

Use that $e^{a+b} = e^a e^b$ to separate constant term

$$S = e^{kt^2} e^c$$

Set $A = e^c$

$$S = Ae^{kt^2}$$

Matches required form with $\alpha = k$ and $\beta = 2$ **[1 Mark]**

Exemplar Exam Question Answer

$$S = Ae^{kt^2}$$

(iii) Determine value of A

Initial surface area for $t = 0$ is

$$S = Ae^0$$

$$S = A$$

We are given that the initial radius is r_0, so we need to find A in terms of r_0

$$S = 4\pi r^2$$

$$A = 4\pi {r_0}^2$$

Equation for S now reads

$$S = 4\pi {r_0}^2 e^{kt^2}$$

[1 Mark]

Exemplar Exam Question Answer

$$S = 4\pi r_0^{\,2} e^{kt^2}$$

Calculate surface area at $t = 1$.

We are given that $k = 2.65$ s

We are also given that $r_0 = 1.39 \times 10^{14}$ m

Want to find surface area at time $t = 1$

Substitute values into equation for S and calculate

$$S = 4\pi\left(1.39 \times 10^{14}\right)^2 e^{(2.65)\times 1^2}$$

$$S = 3.44 \times 10^{30}\ \text{m}^2$$

[1 Mark]

Integration Methods

$$\int_a^b \left(u \times \frac{dv}{dx} \right) dx = [u \times v]_a^b - \int_a^b \left(v \times \frac{du}{dx} \right) dx$$

Specification Points - AQA

	Content
H5	Carry out simple cases of integration by substitution and integration by parts; understand these methods as the inverse processes of the chain and product rules respectively. (Integration by substitution includes finding a suitable substitution and is limited to cases where one substitution will lead to a function which can be integrated; integration by parts includes more than one application of the method but excludes reduction formulae).

	Content
H6	Integrate using partial fractions that are linear in the denominator.

Specification Points – OCR A

h) Be able to carry out simple cases of integration by substitution.

Learners should understand the relationship between this method and the chain rule.

Learners will be expected to integrate examples in the form $f'(x)(f(x))^n$*, such as* $(2x+3)^5$ *or* $x(x^2+3)^7$*, either by inspection or substitution.*

Learners will be expected to recognise an integrand of the form $\frac{kf'(x)}{f(x)}$ *such as* $\frac{x^2+x}{2x^3+3x^2-7}$ *or* $\tan x$*.*

Integration by substitution is limited to cases where one substitution will lead to a function which can be integrated. Substitutions may or may not be given.

Learners should be able to find a suitable substitution in integrands such as $\frac{(4x-1)}{(2x+1)^5}$, $\sqrt{9-x^2}$ *or* $\frac{1}{1+\sqrt{x}}$.

i) Be able to carry out simple cases of integration by parts.

Learners should understand the relationship between this method and the product rule.

Integration by parts may include more than one application of the method e.g. $x^2 \sin x$*.*

Learners will be expected to be able to apply integration by parts to the integral of $\ln x$ *and related functions.*

[*Reduction formulae are excluded.*]

j) Be able to integrate functions using partial fractions that have linear terms in the denominator.

i.e. Functions with denominators no more complicated than the forms $(ax+b)(cx+d)^2$ *or* $(ax+b)(cx+d)(ex+f)$.

Specification Points – OCR MEI

c27	Be able to use integration by substitution in cases where the process is the reverse of the chain rule (including finding a suitable substitution).	e.g. $(1+2x)^8$, $x(1+x^2)^8$, xe^{x^2}, $\frac{1}{2x+3}$ Learners can recognise the integral, they need not show all the working for the substitution.
c28	Be able to use integration by substitution in other cases.	Learners will be expected to find a suitable substitution in simple cases e.g. $\frac{x}{(x+1)^3}$.
c29	Be able to use the method of integration by parts in simple cases.	Includes cases where the process is the reverse of the product rule. e.g. xe^x. More than one application of the method may be required. Includes being able to apply integration by parts to $\ln x$.

Specification Points - Edexcel

8.5	Carry out simple cases of integration by substitution and integration by parts; understand these methods as the inverse processes of the chain and product rules respectively (Integration by substitution includes finding a suitable substitution and is limited to cases where one substitution will lead to a function which can be integrated; integration by parts includes more than one application of the method but excludes reduction formulae.)	Students should recognise integrals of the form $\int \frac{f'(x)}{f(x)} dx = \ln f(x) + c$. The integral $\int \ln x \, dx$ is required Integration by substitution includes finding a suitable substitution and is limited to cases where one substitution will lead to a function which can be integrated; integration by parts includes more than one application of the method but excludes reduction formulae.

8.6	Integrate using partial fractions that are linear in the denominator.	Integration of rational expressions such as those arising from partial fractions, e.g. $\frac{2}{3x+5}$ Note that the integration of other rational expressions, such as $\frac{x}{x^2+5}$ and $\frac{2}{(2x-1)^4}$ is also required (see previous paragraph).

Partial Fractions

Partial fractions can **transform** an **expression** which is difficult to **integrate** to several **expressions** which are easier to **integrate**.

$$\int \frac{mx+n}{(x+a)(x+b)(x+c)}dx \longrightarrow \int \frac{A}{(x+a)}dx + \int \frac{B}{(x+b)}dx + \int \frac{C}{(x+c)}dx$$

To **convert** an **algebraic fraction** to **partial fraction form**:

- **Factorise** the **denominator**.
- **Equate** to **partial fraction form**.

$$\frac{mx+n}{(x+a)(x+b)} = \frac{A}{(x+a)} + \frac{B}{(x+b)}$$

Partial Fractions

$$\frac{mx+n}{(x+a)(x+b)} = \frac{A}{(x+a)} + \frac{B}{(x+b)}$$

- **Multiply** both **sides** by the **denominator factors**.

$$mx+n = A(x+b) + B(x+a)$$

- **Select value** of x to **substitute** to **eliminate** A.

$x=-b$: $\quad f(-b) = B \times (-b+a) \longrightarrow B = \dfrac{m(-b)+n}{(-b+a)}$

- **Select value** of x to **substitute** to **eliminate** B.

$x=-a$: $\quad f(-a) = A \times (-a+b) \longrightarrow A = \dfrac{m(-a)+n}{(-a+b)}$

Partial Fractions

- When the **denominator** contains a **repeated** (**squared**) **factor** which cannot be **factorised**, this **factor** appears **twice** in the **partial fractions**:

$$\frac{1}{(x+2)^2(x-3)} = \frac{A}{(x+2)} + \frac{B}{(x+2)^2} + \frac{C}{(x-3)}$$

- When the **highest power** of the **numerator** is more than the **highest power** of the **denominator**, we have first use **algebraic division**:

$$\frac{4x^3+10x+4}{x(2x+1)} = \qquad \frac{\quad}{x(2x+1)}$$

Exemplar Exam Question

Remember **constant of integration**

1) Resolve the indefinite integral

$$\int \frac{1}{x^3 - 4x}\,dx$$

Too complicated to integrate, can we use **partial fractions?**

Expect to see **logarithms** or **reciprocals** in solution

[4 marks]

4 steps, a fair amount of work to be done to get result.

Exemplar Exam Question Answer

Let $f(x) = \frac{1}{x^3 - 4x}$

Denominator can be factorised and $f(x)$ converted to partial fractions.

Factorise denominator of fraction

Take out factor of x

$$x^3 - 4x = x(x^2 - 4)$$

Recognise squared bracket and factorise again

$$x(x^2 - 4) = x(x + 2)(x - 2)$$

Exemplar Exam Question Answer

$$f(x) = \frac{1}{x(x+2)(x-2)}$$

Split into partial fractions.

Highest power of numerator is less than denominator.

So fraction can break down into

$$\frac{1}{x(x+2)(x-2)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-2}$$

Multiply by $x(x+2)(x-2)$ to get

$$1 = A(x+2)(x-2) + Bx(x-2) + Cx(x+2)$$

[1 Mark]

Exemplar Exam Question Answer

$$1 = A(x+2)(x-2) + B(x)(x-2) + C(x)(x+2)$$

Substitute $x = 0, 2$ and -2 to eliminate terms to find A, B and C

$x = 0$ $\quad 1 = A(0+2)(0-2) + B(0)(0-2) + C(0)(0+2)$

$$1 = -4A + 0 + 0$$

$$A = -\frac{1}{4}$$

[1 Mark]

Exemplar Exam Question Answer

$$1 = -\frac{1}{4}(x+2)(x-2) + B(x)(x-2) + C(x)(x+2)$$

Substitute $x = 0, 2$ and -2 to eliminate terms to find A, B and C

$x = 2$

$$1 = -\frac{1}{4}(2+2)(2-2) + B(2)(2-2) + C(2)(2+2)$$

$$1 = 0 + 0 + 8C$$

$$C = \frac{1}{8}$$

Exemplar Exam Question Answer

$$1 = -\frac{1}{4}(x+2)(x-2) + B(x)(x-2) + \frac{1}{8}(x)(x+2)$$

Substitute $x = 0, 2$ and -2 to eliminate terms to find A, B and C

$x = -2$ $\quad 1 = -\frac{1}{4}(-2+2)(-2-2) + B(-2)(-2-2) + \frac{1}{8}(-2)(-2+2)$

$$1 = 0 + 8B + 0$$

$$B = \frac{1}{8}$$

[1 Mark]

Exemplar Exam Question Answer

$$\int -\frac{1}{4x} + \frac{1}{8(x+2)} + \frac{1}{8(x-2)}\, dx$$

Integrate function in partial fraction form.

All terms are logarithmic terms $\int \frac{a}{x}\, dx = a \ln x$

So integral has general solution

$$-\frac{1}{4}\ln x + \frac{1}{8}\ln(x+2) + \frac{1}{8}\ln(x-2) + c$$

Where c is an integration constant.

[1 Mark]

Integration by Substitution

Changing the **variable** of an **integral** can sometimes **simplify integration**.

$$I = \int_0^2 x(x+1)^3\,dx \qquad u = x + 1$$

- In your **exam** you will often be **instructed** which **substitution** to use.

1. Replace the x-variable in the **integral** with the u **substitution**.

Integration by Substitution

2. **Replace** the dx with du.

3. **Change** the limits of the **integral**.

4. **Carry out** the **integration**.

Remember to **find du** for substitution

Exemplar Exam Question

What **trig identities** would be helpful in this question?

1) Use the substitution $x = \sin^2 u$ to find the general solution of the integral in terms of x:

$$\int \frac{1}{\sqrt{x(1-x)}}\,dx$$

Don't forget **constant!**

4 marks, 4 main steps to calculation

[4 marks]

Exemplar Exam Question Answer

$$\int \frac{1}{\sqrt{x(1-x)}}\,dx \qquad x = \sin^2 u$$

Apply substitution $x = \sin^2 u$ to integral:

$$\int \frac{1}{\sqrt{\sin^2 u\,(1-\sin^2 u)}}\,dx$$

Exemplar Exam Question Answer

$$\int \frac{1}{\sqrt{\sin^2 u\,(1-\sin^2 u)}}\,dx \qquad x = \sin^2 u$$

Determine dx in terms of du:

Differentiate x with respect to u

$$\frac{dx}{du} = 2\sin u \cos u$$

Multiply by du to find relation between dx and du

$$dx = 2\sin u \cos u \;\, du$$

[1 Mark]

Exemplar Exam Question Answer

$$\int \frac{1}{\sqrt{x(1-x)}}\,dx$$

$$x = \sin^2 u$$
$$dx = 2\sin u \cos u \;du$$

Apply substitution $dx = 2\sin u \cos u \;du$ to integral:

$$\int \frac{1}{\sqrt{\sin^2 u\,(1-\sin^2 u)}}(2\sin u \cos u)\,du$$

[1 Mark]

Exemplar Exam Question Answer

$$\int \frac{1}{\sqrt{\sin^2 u\,(1-\sin^2 u)}}(2\sin u\cos u)\,du$$

Rearrange and simplify integral

Know from trigonometry that $\sin^2\theta + \cos^2\theta = 1$

So can rewrite the $1-\sin^2 u$ term to get

$$\int \frac{1}{\sqrt{\sin^2 u\cos^2 u}}(2\sin u\cos u)\,du$$

Exemplar Exam Question Answer

$$\int \frac{1}{\sqrt{\sin^2 u \cos^2 u}} (2 \sin u \cos u)\, du$$

Square root is now easy to resolve, can also move other terms to give

$$\int \frac{2 \sin u \cos u}{\sin u \cos u}\, du$$

Trig functions in numerator and denominator cancel so integral simplifies to

$$\int 2\, du$$

[1 Mark]

Exemplar Exam Question Answer

$$\int \frac{1}{\sqrt{x(1-x)}}\,dx = \int 2\,\mathrm{d}u$$

Resolve integral

Integral is just a constant, so can easily resolve to give

$$\int \frac{1}{\sqrt{x(1-x)}}\,dx = 2u + \mathrm{c}$$

Exemplar Exam Question Answer

$$\int \frac{1}{\sqrt{x(1-x)}}\,dx = 2u + c$$

Convert answer to form only containing terms in x.

$x = \sin^2 u \qquad \Rightarrow u = \arcsin\left(\pm\sqrt{x}\right)$

$$\int \frac{1}{\sqrt{x(1-x)}}\,dx = 2\arcsin\left(\pm\sqrt{x}\right) + c$$

[1 Mark]

Integration by Parts

To **integrate products** of **functions** we can use **integration by parts**:

$$\int_a^b \left(u \times \frac{dv}{dx}\right) dx = [u \times v]_a^b - \int_a^b \left(v \times \frac{du}{dx}\right) dx$$

Integration by Parts

$$\int_a^b \left(u \times \frac{dv}{dx}\right) dx = [u \times v]_a^b - \int_a^b \left(v \times \frac{du}{dx}\right) dx$$

$$\int xe^{2x}\, dx$$

$u =$ $\frac{du}{dx} =$ $\frac{dv}{dx} =$ $v =$

- **Select** u to be the **function** which becomes **simpler** when **differentiated**.
- **Select** $\frac{dv}{dx}$ to be the **other function** which must be **integrated**.

Exemplar Exam Question

Keep your **answer in terms of e**

Recall how to **calculate** the **area under a curve**.

1) Determine the exact area under the curve given by

$$y = \sqrt{x}\ln x$$

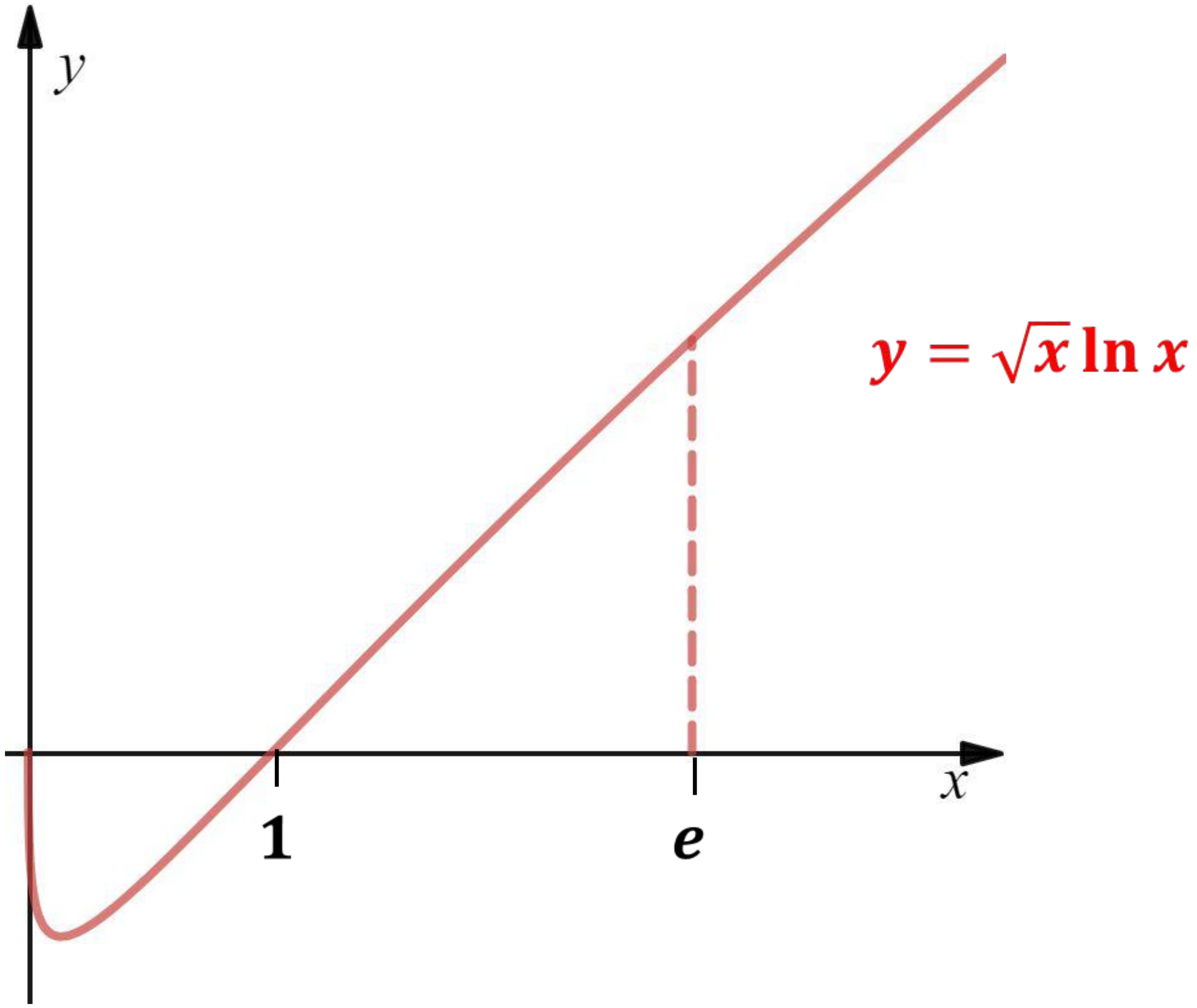
in the domain $1 \le x \le e$.

The **product** of **2 functions** means we will have to use **integration by parts**.

[5 marks]

What are **limits for integral**?

5 steps to question


y
x
1
e
$y = \sqrt{x}\ln x$

Exemplar Exam Question Answer

$$y = \sqrt{x}\ln x$$

Construct integral

Exact area under the curve is given by its integral.

Set limits to be the boundaries of the given domain.

$$A = \int_1^e \sqrt{x}\ln x \, dx$$

Exemplar Exam Question Answer

$$A = \int_1^e \sqrt{x} \ln x \; dx$$

Integral cannot be solved in current form. Integrate by parts to simplify

$$\int_a^b \left(u \times \frac{dv}{dx}\right) dx = [u \times v]_a^b - \int_a^b \left(v \times \frac{du}{dx}\right) dx$$

Identify functions u and v and determine $\frac{du}{dx}$ and $\frac{dv}{dx}$.

$u = \ln x \qquad \frac{du}{dx} = \frac{1}{x}$

$\frac{dv}{dx} = \sqrt{x} \qquad v = \frac{2}{3}x^{\frac{3}{2}}$

[1 Mark]

Exemplar Exam Question Answer

$$A = \int_1^e \sqrt{x} \ln x \; dx$$

$$u = \ln x \qquad \frac{du}{dx} = \frac{1}{x}$$

$$\frac{dv}{dx} = \sqrt{x} \qquad v = \frac{2}{3}x^{\frac{3}{2}}$$

Apply to formula

$$\int_a^b \left(u \times \frac{dv}{dx}\right) dx = [u \times v]_a^b - \int_a^b \left(v \times \frac{du}{dx}\right) dx$$

To find

$$A = \left[\ln x \times \frac{2}{3}x^{\frac{3}{2}}\right]_1^e - \int_1^e \frac{2}{3}x^{\frac{3}{2}} \times \frac{1}{x} dx$$

[1 Mark]

Exemplar Exam Question Answer

$$A = \left[\frac{2}{3}x^{\frac{3}{2}}\ln x\right]_1^e - \int_1^e \frac{2}{3}x^{\frac{3}{2}} \times \frac{1}{x}\,dx$$

Integrate $\int_1^e \frac{2}{3}x^{\frac{3}{2}} \times \frac{1}{x}\,dx$ using index laws:

$$\int_1^e \frac{2}{3}x^{\frac{3}{2}} \times \frac{1}{x}\,dx = \frac{2}{3}\int_1^e x^{\frac{1}{2}}\,dx \quad = \left[\frac{2}{3} \times \frac{2}{3}x^{\frac{3}{2}}\right]_1^e = \left[\frac{4}{9}x^{\frac{3}{2}}\right]_1^e$$

$$\Rightarrow A = \left[\frac{2}{3}x^{\frac{3}{2}}\ln x - \frac{4}{9}x^{\frac{3}{2}}\right]_1^e$$

[1 Mark]

Exemplar Exam Question Answer

$$A = \left[\frac{2}{3}x^{\frac{3}{2}}\ln x - \frac{4}{9}x^{\frac{3}{2}}\right]_1^e$$

Substitute bounds to find area:

$$A = \left(\frac{2}{3}e^{\frac{3}{2}}\ln e - \frac{4}{9}e^{\frac{3}{2}}\right) - \left(\frac{2}{3}1^{\frac{3}{2}}\ln 1 - \frac{4}{9}1^{\frac{3}{2}}\right)$$

$$A = \left(\frac{2}{3}e^{\frac{3}{2}} - \frac{4}{9}e^{\frac{3}{2}}\right) - \left(0 - \frac{4}{9}\right)$$

$$A = \frac{2}{9}e^{\frac{3}{2}} + \frac{4}{9}$$

[1 Mark]

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Exam Question

1. (i) Using the substitution $u = x - 1$, show that

$$\int \left(\frac{x}{x-1}\right)^2 dx = 2\ln(x-1) + \frac{x^2-2x}{1-x} + c$$

Where c is an integration constant

[3 marks]

(ii) Hence or otherwise, show that general solution to the differential equation

$$\frac{dy}{dx} - y = \frac{2y}{x-1} + \frac{y^3}{(xy-y)^2}$$

Can be given as $y = Af(x)e^{g(x)}$ for some expressions $f(x)$ and $g(x)$.

[5 marks]

Exam Question Answer

(i) Rewrite dx in terms of du

$$u = x - 1 \Rightarrow du = dx$$

Substitute for x and dx

$$\int \left(\frac{x}{x-1}\right)^2 dx = \int \left(\frac{u+1}{u}\right)^2 du$$

$$= \int \frac{u^2+2u+1}{u^2} du$$

[1 Mark]

Exam Question Answer

Resolve integral

$$\int \frac{u^2+2u+1}{u^2}\,du = \int 1 + \frac{2}{u} + \frac{1}{u^2}\,du$$

$$\int \frac{u^2+2u+1}{u^2}\,du = u + 2\ln u - \frac{1}{u} + c$$

[1 Mark]

Substitute back x and simplify fraction

$$u + 2\ln u - \frac{1}{u} + c = (x-1) + 2\ln(x-1) - \frac{1}{x-1} + c$$

$$= \frac{(x-1)^2-1}{x-1} + 2\ln(x-1) + c$$

$$= \frac{x^2-2x}{x-1} + 2\ln(x-1) + c$$

[1 Mark]

Exam Question Answer

(ii) Separate $\frac{dy}{dx}$ term

$$\frac{dy}{dx} = y + \frac{2y}{x-1} + \frac{y^3}{(xy-y)^2}$$

Simplify R.H.S

$$\frac{dy}{dx} = y + \frac{2y}{x-1} + \frac{y^3}{y^2(x-1)^2}$$

$$= y + \frac{2y}{x-1} + \frac{y}{(x-1)^2}$$

$$= y\left[1 + \frac{2}{x-1} + \frac{1}{(x-1)^2}\right]$$

[1 Mark]

Exam Question Answer

$$\frac{dy}{dx} = y\left[1 + \frac{2}{x-1} + \frac{1}{(x-1)^2}\right]$$

$$= y\left[\frac{(x-1)^2+2(x-1)+1}{(x-1)^2}\right]$$

$$= y\left[\frac{x^2-2x+1+2x-2+1}{(x-1)^2}\right]$$

$$= y\left[\frac{x^2}{(x-1)^2}\right]$$

$$= y\left(\frac{x}{x-1}\right)^2$$

[1 Mark]

Exam Question Answer

Divide through by y and integrate using solution to part (i)

$$\frac{1}{y}\frac{dy}{dx} = \left(\frac{x}{x-1}\right)^2$$

$$\int \frac{1}{y}\, dy = \int \left(\frac{x}{x-1}\right)^2\, dx$$

$$\ln y = \frac{x^2-2x}{x-1} + 2\ln(x-1) + c$$

$$y = e^{\frac{x^2-2x}{x-1}+2\ln(x-1)+c}$$

$$y = e^{\frac{x^2-2x}{x-1}} \times e^{2\ln(x-1)} \times e^c$$

[1 Mark]

Exam Question Answer

Simplify exponential

Let $e^c = A$

$$e^{2\ln(x-1)} = e^{\ln(x-1)^2} = (x-1)^2$$

[1 Mark]

$$\Rightarrow y = e^{\frac{x^2-2x}{x-1}} e^{2\ln(x-1)} e^c$$

$$= e^{\frac{x^2-2x}{x-1}} \times (x-1)^2 \times A = A(x-1)^2 e^{\frac{x^2-2x}{x-1}}$$

$$\Rightarrow f(x) = (x-1)^2,\ g(x) = \frac{x^2-2x}{x-1}$$

[1 Mark]

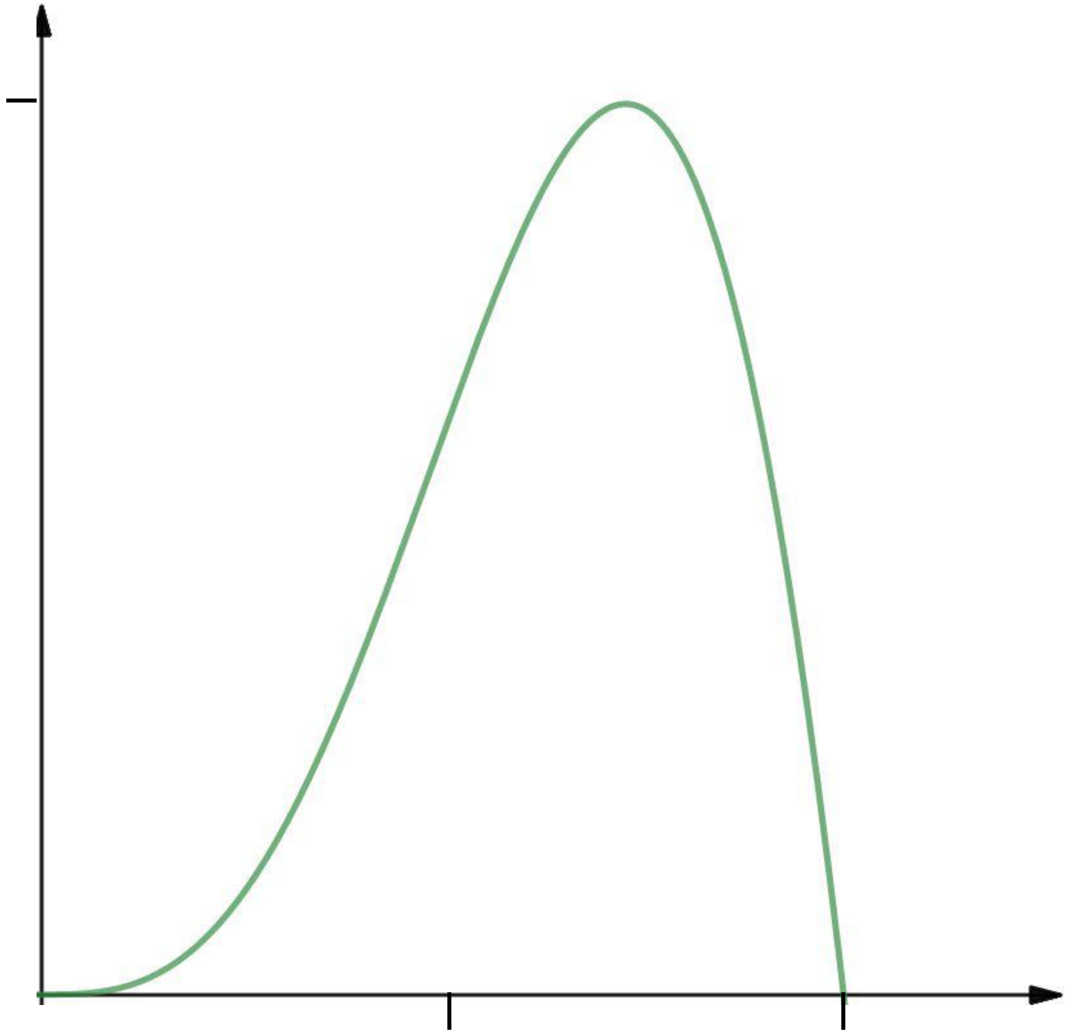
Exam Question

2. Determine the exact area under the curve given by

$$y = x^2 \sin 2x$$

in the domain $0 \leq x \leq \frac{\pi}{2}$.

[5 marks]



Exam Question Answer

Construct integral

Exact area under the curve is given by its integral.

Set limits to be the boundaries of the given domain.

Exam Question Answer

Integral cannot be solved in current form. Integrate by parts to simplify

Exam Question Answer

Apply to formula

$$\int_a^b \left(u \times \frac{dv}{dx}\right) dx = [u \times v]_a^b - \int_a^b \left(v \times \frac{du}{dx}\right) dx$$

To find

$$A = \left[x^2 \times -\frac{1}{2}\cos 2x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} -\frac{1}{2}\cos 2x \times 2x \, dx$$

[1 Mark]

Exam Question Answer

Exam Question Answer

Apply to formula

[1 Mark]

Exam Question Answer

Solve remaining integral.

Last integral can be solved normally:

[1 Mark]

Exam Question Answer

Plug in the limits for the integral.

[1 Mark]

Exam Question Answer

[1 Mark]